ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology and its Applications 153 (2006) 2528–2550
Topology and its Applications
www.elsevier.com/locate/topol

On C^*-algebras and K-theory for infinite-dimensional Fredholm manifolds

Dorin Dumitraşcu [a], Jody Trout [b,*,1]

[a] 617 N. Santa Rita, University of Arizona, Tucson, AZ 85721-0089, USA
[b] 6188 Bradley Hall, Dartmouth College, Hanover, NH 03755, USA

Received 13 November 2004; accepted 28 September 2005

Abstract

Let M be a smooth Fredholm manifold modeled on a separable infinite-dimensional Euclidean space $\mathcal{E}$ with Riemannian metric g. Given an augmented Fredholm filtration $\mathcal{F}$ of M by finite-dimensional submanifolds $\{M_n\}_{n=k}^{\infty}$, we associate to the triple $(M, g, \mathcal{F})$ a non-commutative direct limit C^*-algebra

$$\mathcal{A}(M, g, \mathcal{F}) = \varinjlim \mathcal{A}(M_n)$$

that can play the role of the algebra of functions vanishing at infinity on the non-locally compact space M. The C^*-algebra $\mathcal{A}(\mathcal{E})$, as constructed by Higson–Kasparov–Trout for their Bott periodicity theorem, is isomorphic to our construction when $M = \mathcal{E}$. If M has an oriented Spin_q-structure $(1 \leqslant q \leqslant \infty)$, then the K-theory of this C^*-algebra is the same (with dimension shift) as the topological K-theory of M defined by Mukherjea. Furthermore, there is a Poincaré duality isomorphism of this K-theory of M with the compactly supported K-homology of M, just as in the finite-dimensional spin setting.

MSC: 19M05; 46L85; 46T05; 55N15; 57N20; 58B15

Keywords: C^*-algebra; Fredholm manifold; Direct limit; K-theory; K-homology; Poincaré duality

1. Introduction

Infinite-dimensional Hilbert manifolds have been studied since the 1960's, with main applications in infinite-dimensional differential topology, global analysis, non-linear PDEs, and other areas. This paper is concerned with constructing C^*-algebras and computing the K-theory for a particular class of infinite-dimensional Hilbert manifolds, namely *Fredholm manifolds* [18,20,21]. This is part of a research program to introduce concepts and techniques from Alain Connes' non-commutative geometry [11], e.g., C^*-algebras, K-theory, cyclic (co)homology, and spectral triples, into the study of Fredholm manifolds.

* Corresponding author.
E-mail address: jody.trout@dartmouth.edu (J. Trout).
1 The second author was partially supported by NSF Grant DMS-0071120.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2005.09.008

But first, let us review the finite-dimensional case. Given M a *finite-dimensional* Riemannian manifold, let $C_0(M)$ be the commutative C^*-algebra of all continuous complex-valued functions which vanish at infinity on M. This C^*-algebra categorically encodes the topological properties of M [46] and, by the Serre–Swan theorem, plays a dual role in the K-theory of M:

$$K^j(M) \cong K_j\big(C_0(M)\big), \quad j = 0, 1,$$

where $K^j(M)$ is the (reduced) topological K-theory of M [3]. Furthermore, if M has a spin (or spin^c) structure [33], there is a Poincaré duality isomorphism [26,43]:

$$K^{n-j}(M) \cong K_j^c(M), \quad j = 0, 1,$$

where $K_j^c(M)$ denotes the dual (compactly supported) K-homology of M and n is the dimension of M.

The other C^*-algebra for a finite-dimensional M is non-commutative and constructed using the Riemannian metric g. For each $x \in M$, the tangent space T_xM of M is a finite-dimensional Euclidean space with inner product g_x. Thus, we can form the complex Clifford algebra $\mathrm{Cliff}(T_xM, g_x)$ (see Section 2). It has a canonical structure as a finite-dimensional $\mathbb{Z}_2$-graded C^*-algebra. The family of C^*-algebras $\{\mathrm{Cliff}(T_xM, g_x)\}_{x\in M}$ naturally forms a $\mathbb{Z}_2$-graded, C^*-algebra vector bundle $\mathrm{Cliff}(TM) \to M$, called the *Clifford algebra bundle* of M [4]. We then can define

$$\mathcal{C}(M) = C_0\big(M, \mathrm{Cliff}(TM)\big)$$

to be the C^*-algebra of continuous sections of the Clifford algebra bundle of M vanishing at infinity. This C^*-algebra was used by Kasparov [29] in studying the Novikov Conjecture, where he used the notation $\mathcal{C}_\tau(M)$. If M is even-dimensional and has a spin structure (or, more generally, a spin^c-structure) then this C^*-algebra is Morita equivalent to $C_0(M)$. (In general, $\mathcal{C}(M)$ is Morita equivalent to $C_0(TM)$.) By the Morita invariance of K-theory, it follows that

$$K_j\big(\mathcal{C}(M)\big) \cong K_j\big(C_0(M)\big) \cong K^j(M), \quad j = 0, 1.$$

For M odd-dimensional and spin, this is more complicated. (See Proposition 5.14.)

If M is an *infinite-dimensional* Hilbert manifold [32], modeled on a separable infinite-dimensional Euclidean (i.e., real Hilbert) space $\mathcal{E}$, then these two constructions do not work. Both fail since compact subsets of $M = \mathcal{E}$ are "thin", i.e., have empty interior. Thus, $C_0(\mathcal{E}) = \{0\}$ since there are no compactly supported continuous functions on $\mathcal{E}$ which are non-zero. However, the Clifford C^*-algebra has been generalized by Higson–Kasparov–Trout [25] to the case $M = \mathcal{E}$, by a direct limit construction that exploits an important property of Clifford algebras with respect to orthogonal sums (see Eq. (2)). The component C^*-algebras in the direct limit are given by

$$\mathcal{A}\big(E^a\big) = C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathcal{C}\big(E^a\big) \cong C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} C_0\big(E^a, \mathrm{Cliff}\big(E^a\big)\big)$$

where $\widehat{\otimes}$ denote the $\mathbb{Z}_2$-graded tensor product [6] and $C_0(\mathbb{R})$ is graded by even and odd functions. Since the map $E^a \mapsto \mathcal{A}(E^a)$ is functorial with respect to inclusions of finite-dimensional subspaces, one can construct a non-commutative direct limit C^*-algebra (in the better notation of [24]):

$$\mathcal{A}(\mathcal{E}) = \varinjlim_{E^a \subset \mathcal{E}} \mathcal{A}\big(E^a\big)$$

where the direct limit is taken over *all finite-dimensional subspaces* $E^a \subset \mathcal{E}$. (See Example 4.3 for more on this construction and how it fits into our theory.) This C^*-algebra was used to prove an equivariant Bott periodicity theorem for infinite-dimensional Euclidean spaces [25] and has had applications to proving cases of the Novikov Conjecture and, more generally, the Baum–Connes Conjecture [24,49].

Now, suppose the Hilbert manifold M is fibered as the total space of a smooth infinite rank Euclidean vector bundle $p : F \to X$, with fiber $\mathcal{E}$ and compatible affine connection ∇, over a finite-dimensional Riemannian manifold X. Let $p_a : F^a \to X$ be a *finite rank* subbundle of F. Using the connection ∇ and the metrics on F and X, we can give the total space F_a a canonical structure of a Riemannian manifold and define the component C^*-algebra

$$\mathcal{A}\big(F^a\big) = C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathcal{C}\big(F^a\big) \cong C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} C_0\big(F^a, \mathrm{Cliff}\big(TF^a\big)\big).$$

Since the map $F^a \mapsto \mathcal{A}(F^a)$ is functorial with respect to inclusions of finite-dimensional subbundles [45], we can then construct a direct limit C^*-algebra:

$$\mathcal{A}(F, \nabla) = \varinjlim_{F^a \subset F} \mathcal{A}\big(F^a\big)$$

where the direct limit is taken over *all finite rank subbundles* $p_a : F^a \to X$ of F. Trout [45] used this C^*-algebra to prove an equivariant Thom isomorphism theorem for infinite rank Euclidean bundles, which reduces to the Higson–Kasparov–Trout Bott periodicity theorem when the base manifold X is a point.

For a more general *curved* Hilbert manifold M, with Riemannian metric g, there does not seem to be a natural generalization of the previous constructions. Based on the above, one would be tempted to construct a direct limit C^*-algebra

$$\text{“}\mathcal{A}(M) = \varinjlim_{M_a \subset M} \mathcal{A}(M_a)\text{”}$$

where the component C^*-algebras should be given by

$$\mathcal{A}(M_a) = C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathcal{C}(M_a)$$

and the direct limit is taken over *all finite-dimensional submanifolds* $M_a \subset M$. The problem is that, even though the component C^*-algebras have many functoriality properties (as discussed in Section 2), if we are given smooth (isometric) inclusions

$$M_a \subset M_b \subset M_c$$

of finite-dimensional submanifolds of M, there is no obvious way to define a commuting diagram (as there is in the Bott periodicity and Thom isomorphism cases)

$$\begin{tikzcd} & \mathcal{A}(M_b) \arrow[dr] & \\ \mathcal{A}(M_a) \arrow[ur] \arrow[rr] & & \mathcal{A}(M_c) \end{tikzcd} \tag{1}$$

needed to construct the corresponding direct limit.

However, if the Hilbert manifold M has a *Fredholm structure*, then we can construct a direct limit C^*-algebra by choosing an appropriate *countable sequence* $\{M_n\}_{n=k}^{\infty}$ of expanding, topologically closed, finite-dimensional submanifolds of $\dim(M_n) = n$. The sequence $\{M_n\}_{n=k}^{\infty}$ is called a *Fredholm filtration* of M. (See Section 3 for the geometric definitions and details.) The countability of this sequence of submanifolds clearly simplifies the direct limit construction since only each “Gysin” map $\mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$ needs to be constructed, which will require some non-trivial geometry (i.e., connections and normal bundles.)

Equip the Riemannian–Fredholm manifold (M, g) with an *augmented* Fredholm filtration $\mathcal{F} = (M_n, U_n)_{n=k}^{\infty}$ (as in Definition 3.9) where U_n is a total open tubular neighborhood of $M_n \hookrightarrow M_{n+1}$. Section 4 contains the construction of a non-commutative direct limit C^*-algebra for the triple $(M, g, \mathcal{F})$:

$$\mathcal{A}(M, g, \mathcal{F}) = \varinjlim \mathcal{A}(M_n)$$

that can play the role of the algebra of functions vanishing at infinity on M.

Using ideas of Mukherjea [34,35] to associate cohomology functors to Fredholm manifolds via Fredholm filtrations, the *topological K-theory groups* of $(M, \mathcal{F})$ are defined as the direct limit:

$$K^{\infty-j}(M, \mathcal{F}) = \varinjlim K^{n-j}(M_n), \quad j = 0, 1,$$

where the connecting map $K^{n-j}(M_n) \to K^{(n+1)-j}(M_{n+1})$ is the Gysin (or shriek) map (Definition 5.1) of the embedding $M_n \hookrightarrow M_{n+1}$, and the inspiration for our connecting map $\mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$. Note that this definition does, in general, depend on the choice of Fredholm filtration, since the sequence $\{M_n\}_{n=k}^{\infty}$ may not be K-orientable [17,12].

But, using appropriate notions of Spin_q-structures (see Section 5.2) for Riemannian–Fredholm manifolds, originally investigated by Anastasiei [2] and de la Harpe [14], the following Serre–Swan and Poincaré duality isomorphism theorem (combining Theorems 5.13 and 5.19) is obtained:

Theorem 1.1. *Let (M, g) be a smooth Fredholm manifold with oriented Riemannian q-structure $(1 \leqslant q \leqslant \infty)$. If M has a* Spin_q*-structure then there are isomorphisms*

$$K^{\infty-j}(M, \mathcal{F}) \cong K_{j+1}\big(\mathcal{A}(M, g, \mathcal{F})\big) \cong K_j^c(M), \quad j = 0, 1,$$

where $\mathcal{F} = (M_n, U_n)_{n=k}^{\infty}$ is any *augmented Fredholm filtration of M.*

Thus, the K-theory groups of $(M, \mathcal{F})$ and of the C^*-algebra $\mathcal{A}(M, g, \mathcal{F})$ do not depend on the choice of the Riemannian metric g or the (augmented) Fredholm filtration $\mathcal{F}$. The dimension shift and the relation with Poincaré duality for finite-dimensional spin manifolds then justifies our interpretation of $\mathcal{A}(M, g, \mathcal{F})$ as an appropriate non-commutative (suspension of the) "algebra of functions vanishing at infinity" on M.

Finally, it should be noted that, given a Fredholm filtration $\{M_n\}_{n=k}^{\infty}$ of M, we can also naturally associate an *inverse* limit algebra, called by Phillips [37] a σ-C^*-algebra,

$$C_0^{\mathrm{inv}}(M) = \varprojlim C_0(M_n)$$

where the connecting map $C_0(M_{n+1}) \to C_0(M_n)$ is the pullback under the inclusion $M_n \hookrightarrow M_{n+1}$. However, this algebra does *not* have the structure of a C^*-algebra, in general. Moreover, if we try to define the "topological K-theory" of M as the *inverse* limit (using contravariance of topological K-theory)

$$K_{\mathrm{inv}}^{j}(M) = \varprojlim K^j(M_n), \quad j = 0, 1,$$

then we do not get a well-behaved functor. Indeed, as Buhshtaber and Mishchenko have shown, the resulting K-theory sequence of a pair (M, N) is not exact, in general [9,10] even for CW-complexes. Also, K-theory does not behave well with respect to inverse limits since there is a Milnor $\varprojlim^1$-sequence [38, Theorem 3.2]:

$$0 \to \varprojlim{}^1 K^{j+1}(M_n) \to RK_j\big(C_0^{\mathrm{inv}}(M)\big) \to K_{\mathrm{inv}}^{j}(M) \to 0$$

where RK_j is the representable K-theory for σ-C^*-algebras developed by Phillips [38] and Weidner [47]. Hence, there would be no corresponding Serre–Swan duality theorem as in the finite-dimensional category.

2. Clifford C^*-algebras and the Thom $*$-homomorphism

In this section we assemble the constructions and results for finite-dimensional manifolds that are needed to carry out the direct limit construction of the C^*-algebra of an infinite-dimensional Fredholm manifold. All of the manifolds in this section are assumed to be smooth, Hausdorff, paracompact, and finite-dimensional. For a detailed discussion of most of the results in this section, including more proofs, see Section 2 of Trout [45].

Let V be a finite-dimensional Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$. The complex Clifford algebra of V, denoted $\mathrm{Cliff}(V)$, is the universal complex C^*-algebra (with unit) generated by the elements of V such that $v^* = v$ and

$$v \cdot w + w \cdot v = 2\langle v, w \rangle 1$$

for all $v, w \in V$. It has a natural $\mathbb{Z}_2$-grading by declaring that all elements of V have odd degree. The universal property [33,23] of $\mathrm{Cliff}(V)$ is that if $f : V \to A$ is a real linear map of V into a unital complex C^*-algebra A such that

$$f(v)^2 = \langle v, v \rangle 1_A$$

for all $v \in V$ then there is an induced C^*-algebra homomorphism $\tilde{f} : \mathrm{Cliff}(V) \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Cliff}(V) \arrow[dr, "\tilde{f}"] & \\
V \arrow[u, "C"] \arrow[r, "f"'] & A
\end{tikzcd}$$

where we denote by $C : V \hookrightarrow \mathrm{Cliff}(V)$ the canonical inclusion. However, we will usually identify $v = C(v) \in \mathrm{Cliff}(V)$ for all $v \in V$. An important property of these $\mathbb{Z}_2$-graded C^*-algebras is their behavior with respect to orthogonal sums:

$$\mathrm{Cliff}(V \oplus W) \cong \mathrm{Cliff}(V) \mathbin{\widehat{\otimes}} \mathrm{Cliff}(W) \tag{2}$$

where $\widehat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product. (See the books [33,23] for a review of Clifford algebras and Blackadar [6] for a review of graded C^*-algebras.)

Let M_n be a *finite-dimensional* smooth Riemannian manifold of dimension n with Riemannian metric g. Let $TM_n \to M_n$ denote the tangent bundle of M_n. Let $\mathrm{Cliff}(TM_n) \to M_n$ denote the Clifford bundle [4,5] of TM_n, i.e., the bundle of Clifford algebras over M_n whose fiber at $x \in M_n$ is the complex Clifford algebra $\mathrm{Cliff}(T_x M_n)$ of the Euclidean tangent space $T_x M_n$. It has an induced $\mathbb{Z}_2$-graded C^*-algebra bundle structure.

Definition 2.1. [29] Denote by $\mathcal{C}(M_n)$ the C^*-algebra

$$\mathcal{C}(M_n) = C_0\big(M_n, \mathrm{Cliff}(TM_n)\big)$$

of continuous sections of $\mathrm{Cliff}(TM_n)$ which vanish at infinity on M_n, with induced $\mathbb{Z}_2$-grading from $\mathrm{Cliff}(TM_n)$. (Kasparov [29] used the notation $C_\tau(M_n)$.)

For example, if $M_n = V$ is a finite-dimensional Euclidean vector space, then $TM_n \cong V \times V$ and so $\mathcal{C}(M_n) \cong C_0(V, \mathrm{Cliff}(V))$ as in [25, Definition 2.2]. A priori, this C^*-algebra depends on the Riemannian metric g of M_n. However, the universal property of Clifford algebras shows that the C^*-algebra structure on $\mathcal{C}(M_n)$ depends only on the manifold M_n and not the chosen metric g. Indeed, if h is another Riemannian metric on M_n, then $\alpha = \hat{h}^{-1} \circ \hat{g} : TM_n \to TM_n$ is an automorphism of the tangent bundle TM_n, where $\hat{g} : TM_n \to T^*M_n$ is the (co)tangent bundle isomorphism induced by any metric g. It satisfies

$$h\big(\alpha(X), X\big) = g(X, X) \geqslant 0$$

for any vector field X. Thus, α is positive definite with respect to the metric h and so has a positive square root, i.e., a bundle automorphism $\beta : TM_n \to TM_n$ such that

$$h\big(\beta(X), \beta(X)\big) = h\big(\alpha(X), X\big) = g(X, X).$$

If $\mathrm{Cliff}(TM_n, h)$ denotes the Clifford bundle of M_n with respect to the metric h then

$$\beta(X)^2 = g(X, X)1$$

in $\mathrm{Cliff}(TM_n, h)$. By the universal property above (applied to each fiber) β extends to an isomorphism $\tilde{\beta} : \mathrm{Cliff}(TM_n, g) \to \mathrm{Cliff}(TM_n, h)$ of Clifford bundles. (See also [23, Section 9.1].) By taking sections, there is a canonically induced isomorphism

$$\hat{\beta} : \mathcal{C}(M, g) \to \mathcal{C}(M, h)$$

of $\mathbb{Z}_2$-graded C^*-algebras.

Let $C_0(M_n)$ denote the commutative C^*-algebra of continuous complex-valued functions on M_n vanishing at infinity. We always consider $C_0(M_n)$ to be trivially graded. If a $\mathbb{Z}_2$-graded C^*-algebra A is equipped with a (fixed) $*$-homomorphism $\Theta : C_0(M_n) \to Z(M(A))$ that is nondegenerate and has grading degree zero, where $Z(M(A))$ denotes the center of the multiplier algebra of A, then we say that A has a $\mathbb{Z}_2$-graded $C_0(M_n)$-algebra structure [45]. We denote $\Theta(f)a = f \cdot a$ for all $f \in C_0(M_n)$ and $a \in A$. Note that pointwise multiplication $(fs)(x) = f(x)s(x)$, $\forall x \in M_n$, where $f \in C_0(M_n)$ and $s \in \mathcal{C}(M_n)$, determines a nondegenerate $*$-homomorphism $C_0(M_n) \to ZM(\mathcal{C}(M_n))$ into the center of the multiplier algebra of $\mathcal{C}(M_n)$ of grading degree zero. Thus, we have the following.

Corollary 2.2. *The C^*-algebra $\mathcal{C}(M_n)$ has a canonical $\mathbb{Z}_2$-graded $C_0(M_n)$-algebra structure, and up to $\mathbb{Z}_2$-graded isomorphism, is independent of the Riemannian metric on M_n.*

Definition 2.3. Let $\mathcal{S}$ denote the C^*-algebra $C_0(\mathbb{R})$ of continuous complex-valued functions on the real line which vanish at infinity, with $\mathbb{Z}_2$-grading by even and odd functions. If A is any $\mathbb{Z}_2$-graded C^*-algebra then we let $\mathcal{S}A$ be the graded (max) tensor product $\mathcal{S} \mathbin{\widehat{\otimes}} A$. In particular, let

$$\mathcal{A}(M_n) \stackrel{\mathrm{def}}{=} \mathcal{S}\mathcal{C}(M_n) = \mathcal{S} \mathbin{\widehat{\otimes}} C_0\big(M_n, \mathrm{Cliff}(TM_n)\big)$$

which can be viewed as a non-commutative topological suspension of M_n.[2]

[2] Recall that the suspension of a C^*-algebra A is the C^*-algebra $SA = C_0(\mathbb{R}) \otimes A$. In particular, $SC_0(M_n) \cong C_0(\mathbb{R} \times M_n)$ where $\mathbb{R} \times M_n$ is the (reduced) topological suspension of M_n.

The following functoriality result will be used when we identify the total space of the normal bundle of an embedding with an open tubular neighborhood.

Lemma 2.4. [45] *Let $\phi : M_n \to N_n$ be a diffeomorphism of Riemannian manifolds. There is an induced $\mathbb{Z}_2$-graded C^*-algebra isomorphism*

$$\phi_* : \mathcal{A}(M_n) \to \mathcal{A}(N_n).$$

Proof. Let g denote the metric on M_n and h denote the metric on N_n. If $\phi : M_n \to N_n$ is a diffeomorphism, then, using the pullback metric $\phi^*(h)$, we have that

$$\phi : \big(M_n, \phi^*(h)\big) \to (N_n, h)$$

is an *isometry* of Riemannian manifolds, which clearly induces a canonical isomorphism

$$\hat{\phi} : \mathcal{C}\big(M_n, \phi^*(h)\big) \to \mathcal{C}(N_n, h)$$

of $\mathbb{Z}_2$-graded C^*-algebras. By the argument above, we have a canonical isomorphism

$$\hat{\beta} : \mathcal{C}(M_n, g) \to \mathcal{C}\big(M_n, \phi^*(h)\big).$$

Taking the composition and tensoring with the identity of $\mathcal{S}$ gives the required canonical isomorphism

$$\phi_* = id_{\mathcal{S}} \mathbin{\widehat{\otimes}} \big(\hat{\beta} \circ \hat{\phi}\big) : \mathcal{A}(M_n) = \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{C}(M_n) \to \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{C}(N_n) = \mathcal{A}(N_n)$$

of $\mathbb{Z}_2$-graded C^*-algebras. $\square$

The following is an easy functoriality property for open inclusions.

Lemma 2.5. [45] *Let U_n be an open subset of the Riemannian manifold M_n. The inclusion $i : U_n \hookrightarrow M_n$ induces a short exact sequence*

$$0 \to \mathcal{A}(U_n) \xrightarrow{1 \widehat{\otimes} i_*} \mathcal{A}(M_n) \to \mathcal{A}(M_n \setminus U_n) \to 0$$

of C^*-algebras. Thus, $\mathcal{A}(U_n) \lhd \mathcal{A}(M_n)$ as a (two-sided) C^*-ideal.

Let $p : E \to M_n$ be a smooth finite rank Euclidean vector bundle. We will show that there is a natural "Thom" $*$-homomorphism

$$\Psi_p : \mathcal{A}(M_n) \to \mathcal{A}(E),$$

where we consider E as a finite-dimensional manifold with Riemannian structure to be constructed as follows. The main example we have in mind is where $E = \nu M_n$ is the (total space of the) normal bundle of an isometric embedding $M_n \hookrightarrow M_{n+1}$.

Given $p : E \to M_n$, there is a short exact sequence [1,5] of real vector bundles

$$0 \to VE \to TE \xrightarrow{T^*p} p^*TM_n \to 0$$

where the *vertical subbundle* $VE = \ker(T^*p)$ is isomorphic to p^*E. This sequence does *not* have a canonical splitting, in general, but choosing a compatible connection ∇ on E determines an associated vector bundle splitting. Recall that a connection $\nabla : C^\infty(M_n, E) \to C^\infty(M_n, T^*M_n \otimes E)$ on E is *compatible* [5,33] with the bundle metric $(\cdot, \cdot)$ on E if

$$d(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2)$$

for all smooth sections $s_1, s_2 \in C^\infty(M_n, E)$. If $p : E \to M_n$ is equipped with a compatible connection ∇, then we call E an *affine* Euclidean bundle.

Let $\nabla^* : C^\infty(E, p^*E) \to C^\infty(E, T^*E \otimes p^*E)$ denote the pullback of ∇ on the bundle $p^*E \to E$, which is defined by the formula:

$$\nabla^*(fp^*s) = df \otimes p^*s + fp^*(\nabla s)$$

for $f \in C^\infty(M_n)$ and $s \in C^\infty(M_n, E)$. The *tautological section* $\tau \in C^\infty(E, p^*E)$ is the smooth section of $p^*E \to E$ defined by the formula $\tau(e) = (e, e)$ for all $e \in E$. The derivative of τ will be denoted by

$$\omega = \nabla^*\tau \in C^\infty(E, T^*E \otimes p^*E) = \Omega^1(E, p^*E) \cong \Omega^1(E, VE)$$

which is the connection 1-form of ∇ (see [5, Definition 1.10]). The kernel $HE = \ker(\omega) \cong p^*TM_n$ of the connection 1-form ω is the horizontal subbundle of TE which provides a splitting

$$TE = VE \oplus HE \cong p^*E \oplus p^*TM_n.$$

Now give TE the direct sum of the pullback metrics on p^*E and p^*TM_n. This gives E the structure of a Riemannian manifold and makes the splitting of TE orthogonal.

Lemma 2.6. [45] *Let $p : E \to M_n$ be a finite rank affine Euclidean bundle on the Riemannian manifold M_n. There is an induced orthogonal splitting of the exact sequence*

$$0 \to p^*E \to TE \to p^*TM_n \to 0$$

and so there is a canonical isomorphism of Euclidean vector bundles

$$TE \cong p^*E \oplus p^*TM_n$$

where p^*E and p^*TM_n have the pullback metrics. Thus, the manifold E has a canonical Riemannian metric.

Hence, given a compatible connection ∇ on the Euclidean bundle E, we can define the C^*-algebra $\mathcal{C}(E)$ as above using the induced Riemannian structure on the *manifold* E. However, we also have the C^*-algebra $C_0(E, \mathrm{Cliff}(p^*E))$ associated to the pullback bundle $p^*E \to E$.[3] Both $\mathcal{C}(E)$ and $C_0(E, \mathrm{Cliff}(p^*E))$ have natural $C_0(E)$-algebra structures. However, the bundle map $p : E \to M_n$ induces a pullback $*$-homomorphism [41]

$$p^* : C_0(M_n) \to C_b(E) = M\big(C_0(E)\big) : f \mapsto p^*(f) = f \circ p$$

which induces a (graded) $C_0(M_n)$-algebra structure on any (graded) $C_0(E)$-algebra.

Definition 2.7. [45] Let A and B be $\mathbb{Z}_2$-graded $C_0(M_n)$-algebras. The balanced tensor product over M_n, denoted $A \mathbin{\widehat{\otimes}}_{M_n} B$, is the quotient of the maximal graded tensor product $A \mathbin{\widehat{\otimes}} B$ [6] by the ideal J generated by

$$\big\{(f \cdot a) \mathbin{\widehat{\otimes}} b - a \mathbin{\widehat{\otimes}} (f \cdot b):\ a \in A,\ b \in B,\ f \in C_0(M_n)\big\}.$$

For example, $C_0(M_n) \mathbin{\widehat{\otimes}}_{M_n} A \cong A$ via the map induced by $f \mathbin{\widehat{\otimes}} a \mapsto f \cdot a$.

The following is an important result that relates these two C^*-algebras to the C^*-algebra $\mathcal{C}(M_n)$ of the base manifold M_n.

Theorem 2.8. *Let $p : E \to M_n$ be a finite rank affine Euclidean bundle on the Riemannian manifold M_n. There is a natural isomorphism of graded C^*-algebras*

$$\mathcal{C}(E) \cong C_0\big(E, \mathrm{Cliff}(p^*E)\big) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n).$$

Proof. By the previous lemma, there is an induced orthogonal splitting

$$TE = p^*E \oplus p^*TM_n.$$

Thus, we have an induced isomorphism of $\mathbb{Z}_2$-graded Clifford algebra bundles

$$\mathrm{Cliff}(TE) \cong \mathrm{Cliff}(p^*E \oplus p^*TM_n) \cong \mathrm{Cliff}(p^*E) \mathbin{\widehat{\otimes}} p^*\,\mathrm{Cliff}(TM_n). \tag{3}$$

Therefore, by taking sections, we have canonical balanced tensor product isomorphisms (see [45, Proposition A.7])

[3] Note: Although $C_0(E, \mathrm{Cliff}(p^*E)) \cong p^*C_0(M_n, \mathrm{Cliff}(E))$, we will not need this isomorphism.

$$\mathcal{C}(E) \overset{\text{def}}{=} C_0\big(E, \operatorname{Cliff}(TE)\big) \cong C_0\big(E, \operatorname{Cliff}(p^*E) \mathbin{\widehat{\otimes}} p^* \operatorname{Cliff}(TM_n)\big)$$
$$\cong C_0\big(E, \operatorname{Cliff}(p^*E)\big) \mathbin{\widehat{\otimes}}_E C_0\big(E, \operatorname{Cliff}(p^*TM_n)\big).$$

But, we have, using pullbacks along $p : E \to M_n$, that there are canonical pullback isomorphisms (see [45, Proposition A.9])

$$C_0(E, \operatorname{Cliff}\big(p^*TM_n)\big) \cong p^*C_0\big(M, \operatorname{Cliff}(TM_n)\big) = p^*\mathcal{C}(M_n) \overset{\text{def}}{=} C_0(E) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n).$$

Hence, it follows that

$$\mathcal{C}(E) \cong C_0\big(E, \operatorname{Cliff}(p^*E)\big) \mathbin{\widehat{\otimes}}_E C_0\big(E, \operatorname{Cliff}(p^*TM_n)\big) \cong C_0\big(E, \operatorname{Cliff}(p^*E)\big) \mathbin{\widehat{\otimes}}_E C_0(E) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n)$$
$$\cong C_0\big(E, \operatorname{Cliff}(p^*E)\big) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n)$$

using the canonical isomorphism $A \mathbin{\widehat{\otimes}}_E C_0(E) \cong A$ for graded $C_0(E)$-algebras. $\square$

We now wish to define a certain "Thom operator" for the "vertical" algebra $C_0(E, \operatorname{Cliff}(p^*E))$. Associate to the Euclidean bundle E an unbounded section

$$C_E : E \to \operatorname{Cliff}(p^*E) : e \mapsto C_{p(e)}(e)$$

where $C_{p(e)}$ is the Clifford operator on the Euclidean space $E_{p(e)}$ from [25, Definition 2.4]. It is given globally by the composition

$$\begin{tikzcd} E \arrow[r, "\tau"] \arrow[rr, bend right, "C_E"'] & p^*E \arrow[r, "C"] & \operatorname{Cliff}(p^*E) \end{tikzcd}$$

where $\tau \in C^\infty(E, p^*E)$ is the tautological section (see above) and $C : p^*E \hookrightarrow \operatorname{Cliff}(p^*E)$ is the canonical inclusion $C(e_1, e_2) = C_{p(e_1)}(e_2)$. The following is then easy to prove.

Theorem 2.9. *Let E be a finite rank Euclidean bundle on M_n. Multiplication by the section $C_E : E \to \operatorname{Cliff}(p^*E)$ determines a degree one, essentially self-adjoint, unbounded multiplier* (*see* [45, *Definition* A.1]) *of the C^*-algebra $C_0(E, \operatorname{Cliff}(p^*E))$ with domain $C_c(E, \operatorname{Cliff}(p^*E))$.*

We will call C_E the *Thom operator* of $E \to M_n$. Thus, we have a functional calculus homomorphism

$$\mathcal{S} \to M\big(C_0\big(E, \operatorname{Cliff}(p^*E)\big)\big) : f \to f(C_E)$$

from $\mathcal{S}$ to the multiplier algebra of $C_0(E, \operatorname{Cliff}(p^*E))$. Note that $f(C_E)$ goes to zero in the "fiber" directions on E (since $p(e)$ is constant), but is only bounded in the "manifold" directions on E. Indeed, for the generators $f(x) = \exp(-x^2)$ and $g(x) = x\exp(-x^2)$ of $\mathcal{S}$, we have that $f(C_E)$ and $g(C_E)$ are, respectively, multiplication by the following functions on E:

$$f(C_E)(e) = \exp\big(-\|e\|^2\big) \quad \text{and} \quad g(C_E)(e) = e \cdot \exp\big(-\|e\|^2\big), \quad \forall e \in E.$$

Definition 2.10. Let X denote the degree one, essentially self-adjoint, unbounded multiplier of $\mathcal{S}$, with domain the compactly supported functions, given by multiplication by x, i.e., $Xf(x) = xf(x)$ for all $f \in C_c(\mathbb{R})$ and $x \in \mathbb{R}$.

By [45, Lemma A.3], the operator $X \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} C_E$ determines a degree one, essentially self-adjoint, unbounded multiplier of the tensor product

$$\mathcal{S} \mathbin{\widehat{\otimes}} C_0\big(E, \operatorname{Cliff}(p^*E)\big) = \mathcal{S}C_0\big(E, \operatorname{Cliff}(p^*E)\big)$$

with domain $C_c(\mathbb{R}) \mathbin{\widehat{\odot}} C_c(E, \operatorname{Cliff}(p^*E))$. We obtain a functional calculus homomorphism

$$\beta_E : \mathcal{S} \to M\big(\mathcal{S}C_0\big(E, \operatorname{Cliff}(p^*E)\big)\big) : f \mapsto f(X \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} C_E)$$

from $\mathcal{S}$ into the multiplier algebra of $\mathcal{S}C_0(E, \operatorname{Cliff}(p^*E))$. Now we can define our "Thom $*$-homomorphism" for a finite rank affine Euclidean bundle. This will provide part of the connecting map in Section 4 when we define the direct limit C^*-algebra for an infinite-dimensional Riemannian–Fredholm manifold.

Theorem 2.11. *Let $p : E \to M_n$ be a finite rank affine Euclidean bundle on the Riemannian manifold M_n. With respect to the isomorphism*

$$\mathcal{A}(E) \cong \mathcal{S} \mathbin{\widehat{\otimes}} C_0\big(E, \mathrm{Cliff}(p^*E)\big) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n)$$

from Theorem 2.8*, there is a graded $*$-homomorphism*

$$\Psi_p = \beta_E \mathbin{\widehat{\otimes}}_{M_n} \mathrm{id}_{M_n} : \mathcal{A}(M_n) \to \mathcal{A}(E)$$

which on elementary tensors $f \mathbin{\widehat{\otimes}} s \in \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{C}(M_n) = \mathcal{A}(M_n)$ is given by

$$f \mathbin{\widehat{\otimes}} s \mapsto f(X \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} C_E) \mathbin{\widehat{\otimes}}_{M_n} s.$$

Proof. From the discussion above, we have that $\beta_E \mathbin{\widehat{\otimes}}_{M_n} \mathrm{id}_{M_n}$ is the composition

$$\mathcal{A}(M_n) \xrightarrow{\beta_E \widehat{\otimes} \mathrm{id}} M\big(\mathcal{S}C_0\big(E, \mathrm{Cliff}(p^*E)\big)\big) \mathbin{\widehat{\otimes}} \mathcal{C}(M_n) \to M\big(\mathcal{S}C_0\big(E, \mathrm{Cliff}(p^*E)\big)\big) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n).$$

Checking on the generator $f(x) = \exp(-x^2)$ of $\mathcal{S}$, we compute that

$$f(X \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} C_E) \mathbin{\widehat{\otimes}} s = \exp\big(-x^2\big) \mathbin{\widehat{\otimes}} \exp\big(-\|e\|^2\big) \mathbin{\widehat{\otimes}}_{M_n} s \in \mathcal{A}(E).$$

Similarly for $g(x) = x\exp(-x^2)$, we find that

$$g\big(X \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} C_E\big) \mathbin{\widehat{\otimes}}_{M_n} s = x\exp\big(-x^2\big) \mathbin{\widehat{\otimes}} \exp\big(-\|e\|^2\big) \mathbin{\widehat{\otimes}}_{M_n} s + \exp\big(-x^2\big) \mathbin{\widehat{\otimes}} e \cdot \exp\big(-\|e\|^2\big) \mathbin{\widehat{\otimes}}_{M_n} s \in \mathcal{A}(E).$$

It follows that the range of $\Psi_p = \beta_E \mathbin{\widehat{\otimes}}_{M_n} \mathrm{id}_{M_n}$ is in $\mathcal{A}(E)$ as desired. □

Since the space of compatible connections ∇ on $E \to M_n$ is convex, we have the following result.

Proposition 2.12. *Let $p : E \to M_n$ be a smooth finite rank affine Euclidean bundle on the Riemannian manifold M_n. The homotopy class of the $*$-homomorphism $\Psi_p : \mathcal{A}(M_n) \to \mathcal{A}(E)$ is independent of the choice of compatible connection on E.*

Proposition 2.13. *If $p : E = M_n \times V \to M_n$ is a trivial finite rank affine Euclidean bundle* (*with trivial connection $\nabla_0 = d$*) *then we have a $\mathbb{Z}_2$-graded isomorphism*

$$\mathcal{C}(E) \cong \mathcal{C}(V) \mathbin{\widehat{\otimes}} \mathcal{C}(M_n)$$

such that the Thom map has the form

$$\Psi_p \cong \beta_V \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathcal{C}(M_n)} : \mathcal{A}(M_n) = \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{C}(M_n) \to \mathcal{A}(V) \mathbin{\widehat{\otimes}} \mathcal{C}(M_n) \cong \mathcal{A}(E)$$

where $\beta_V : \mathcal{S} \to \mathcal{A}(V) : f \mapsto f(X \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} C_V)$ is the Thom map for $V \to \{0\}$.

Proof. The trivial connection $\nabla_0 = d$ gives the manifold $E = M_n \times V$ the Riemannian metric induced by the isomorphism

$$TE = TM_n \times TV \to M_n \times V = E.$$

The pullback vector bundle $p^*E \to E$ has the form

$$p^*E = (M_n \times V) \times V \to M_n \times V = E$$

and so the Clifford bundle $\mathrm{Cliff}(p^*E) = (M_n \times V) \times \mathrm{Cliff}(V)$, which gives:

$$C_0\big(E, \mathrm{Cliff}(p^*E)\big) = C_0\big(M_n \times V, (M_n \times V) \times \mathrm{Cliff}(V)\big) \cong C_0\big(V, \mathrm{Cliff}(V)\big) \mathbin{\widehat{\otimes}} C_0(M_n).$$

By Theorem 2.8, it follows that

$$\mathcal{C}(E) \cong C_0\big(E, \mathrm{Cliff}(p^*E)\big) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n) \cong C_0\big(V, \mathrm{Cliff}(V)\big) \mathbin{\widehat{\otimes}} C_0(M_n) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n) \cong \mathcal{C}(V) \mathbin{\widehat{\otimes}} \mathcal{C}(M_n),$$

where we used the isomorphism $C_0(M_n) \mathbin{\widehat{\otimes}}_{M_n} \mathcal{C}(M_n) \cong \mathcal{C}(M_n)$. The result now easily follows. □

For example, if $p : E_b \to E_a$ is the orthogonal projection of a finite-dimensional Euclidean vector space E_b onto a linear subspace E_a then $\Psi_p = \beta_{ba}$ is the "Bott homomorphism" from Definition 3.1 of Higson–Kasparov–Trout [25].

3. Fredholm manifolds and filtrations

Fredholm manifolds are a particular case of *Hilbert manifolds*, i.e., manifolds modeled on a separable infinite-dimensional real Hilbert space. Most of the standard constructions from the differential geometry of finite-dimensional manifolds carry on in the infinite-dimensional situation (as reference see Lang's book [32]). All the Hilbert manifolds that we consider in this paper are assumed to be connected, separable, paracompact, Hausdorff, and infinitely smooth.

Let $\mathcal{E}$ be a separable infinite-dimensional Euclidean space, i.e., a real Hilbert space of countably infinite dimension. We will use the following notation: $\mathcal{L}(\mathcal{E})$ denotes the real C^*-algebra of bounded linear operators on $\mathcal{E}$; $\mathcal{F}=\mathcal{F}(\mathcal{E})$ denotes the finite rank operators; $\mathcal{K}=\mathcal{K}(\mathcal{E})$ denotes the closed ideal of compact operators; $\Phi=\Phi(\mathcal{E})$ denotes the Fredholm operators; and $GL(\mathcal{E})$ denotes the Banach–Lie group of units of $\mathcal{L}(\mathcal{E})$, with identity I.

Definition 3.1. A *perturbation class P of $\mathcal{E}$* is a subspace $P=P(\mathcal{E})$ of $\mathcal{L}(\mathcal{E})$ such that:

(1) $\mathcal{F}(\mathcal{E})\subseteq P(\mathcal{E})$,
(2) $P(\mathcal{E})$ is an ideal in $\mathcal{L}(\mathcal{E})$, and
(3) $\Phi(\mathcal{E})+P(\mathcal{E})=\Phi(\mathcal{E})$.

As examples of perturbation classes we have: the finite rank operators $\mathcal{F}(\mathcal{E})$, the compact operators $\mathcal{K}(\mathcal{E})$, or indeed any proper two-sided ideal included in $\mathcal{K}$. For $1\leqslant q<\infty$, let P_q be the perturbation class defined as the closure of $\mathcal{F}(\mathcal{E})$ under the norm

$$\|T\|_q=\big(\mathrm{Trace}(T^*T)^{q/2}\big)^{1/q}. \tag{4}$$

If $q=1$ one obtains the trace-class operators, and if $q=2$ the Hilbert–Schmidt operators. If $q=\infty$, then we set $P_\infty=\mathcal{K}(\mathcal{E})$ with norm $\|T\|_\infty=\|T\|$.

Given a perturbation class P of $\mathcal{E}$, we let

$$GL_P(\mathcal{E})=GL(\mathcal{E})\cap\big(I+P(\mathcal{E})\big)=\big\{T=I+K \mid T\in GL(\mathcal{E}),\ K\in P(\mathcal{E})\big\}.$$

For $1\leqslant q<\infty$, we abbreviate $GL_{P_q}(\mathcal{E})=GL_q(\mathcal{E})$. For $p=\infty$, we abbreviate $GL_{\mathcal{K}(\mathcal{E})}(\mathcal{E})=GL_{\mathcal{K}}(\mathcal{E})$. We topologize $GL_q(\mathcal{E})$ by requiring that the map $GL_q(\mathcal{E})\to\mathcal{O}\subset P_q$, $I+K\mapsto K$, be a homeomorphism, where $\mathcal{O}$ is the set of all K with $I+K$ invertible [36]. In general, $GL_P(\mathcal{E})$ is a normal subgroup of $GL(\mathcal{E})$, but, where $GL(\mathcal{E})$ is contractible (by Kuiper's theorem [31]), $GL_P(\mathcal{E})$ may not be contractible. For example, by Theorem B of Palais [36], we have that

$$\pi_0\big(GL_q(\mathcal{E})\big)=\mathbb{Z}/2\mathbb{Z}$$

for all $1\leqslant q\leqslant\infty$. However, $GL_P(\mathcal{E})$ is not a *closed* subgroup of $GL(\mathcal{E})$ unless $P=P_\infty=\mathcal{K}(\mathcal{E})$.

Definition 3.2. Let M be a Hilbert manifold modeled on $\mathcal{E}$. A *Fredholm structure on M* is an integrable reduction of the principal $GL(\mathcal{E})$-bundle of M to $GL_{\mathcal{K}}(\mathcal{E})$. Equivalently, it is a maximal atlas of M such that the differential of the change of coordinates maps is an element of $GL_{\mathcal{K}}(\mathcal{E})$ at every point. A *Fredholm manifold* is a Hilbert manifold with a specified Fredholm structure.

Since there is a natural inclusion $GL_P(\mathcal{E})\hookrightarrow GL_{\mathcal{K}}(\mathcal{E})$ induced by the inclusion $P(\mathcal{E})\hookrightarrow\mathcal{K}(\mathcal{E})$, if a Hilbert manifold M is equipped with a reduction of it's structure group from $GL(\mathcal{E})$ to $GL_P(\mathcal{E})$ then we can give M a canonical Fredholm structure in the sense of the previous definition. We will make use of this fact when discussing spin structures for Fredholm manifolds in Section 5.

Note. A C^∞-map $f:M\to N$ between Hilbert manifolds is called a *Fredholm map* if, for every $x\in M$, $Df(x):T_xM\to T_{f(x)}N$ is a Fredholm operator. Fredholm manifolds are exactly the manifolds on which Fredholm maps can be constructed. Results of Elworthy and Tromba [21] show that for a Fredholm manifold M there is an index zero (even *bounded* and *proper*) Fredholm map $f:M\to\mathcal{E}$.

The following decomposition theorem is crucial in the study of Fredholm manifolds [34, Theorem 2.2]):

Theorem 3.3. *Let M be a Fredholm manifold. There exists a sequence $\{M_n\}_{n=k}^{\infty}$ of finite dimensional closed submanifolds such that*:

(i) $\dim M_n = n$; $M_n \subset M_{n+1}$;
(ii) *the inclusions $M_n \hookrightarrow M_{n+1}$ and $M_n \hookrightarrow M$ have trivial normal bundles*;
(iii) *$M_\infty = \bigcup_{n\geqslant k} M_n$ is dense in M; and*
(iv) *the natural inclusion map $M_\infty \hookrightarrow M$ is a homotopy equivalence, if M_∞ is given the* direct limit topology.

A sequence $\{M_n\}_{n=k}^{\infty}$ as in the theorem above is called a *Fredholm filtration of M*.

We will now give some examples (and a non-example) of Fredholm manifolds and filtrations.

Examples 3.4.

(i) The Euclidean space $M = \mathcal{E}$ has an obvious Fredholm structure, determined by a single chart $I : \mathcal{E} \to \mathcal{E}$. It is the only possible structure. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{E}$, and E_n be the linear span of $\{e_1, e_2, \ldots, e_n\}$. The sequence $\{E_n\}_n$ is known as a *flag* of $\mathcal{E}$, and it forms a Fredholm filtration.
(ii) The unit sphere of $\mathcal{E}$, $S_{\mathcal{E}} = \{x \in \mathcal{E} \mid \|x\| = 1\}$, gets by restriction from $\mathcal{E}$ a Fredholm structure. As a Fredholm filtration we have
$$S^1 \subset S^2 \subset \cdots \subset S^n \subset \cdots \subset S_{\mathcal{E}}$$
where S^n is the unit n-sphere in the Euclidean space E_{n+1}.
(iii) The following is a *non-example*. The sequence of real projective spaces
$$\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \cdots \subset \mathbb{R}P^n \subset \cdots \subset \mathbb{R}P_{\mathcal{E}}$$
is not a Fredholm filtration of the infinite-dimensional real projective space $\mathbb{R}P_{\mathcal{E}}$ of $\mathcal{E}$, for any choice of Fredholm structure, because the inclusions $\mathbb{R}P^n \subset \mathbb{R}P^{n+1}$ do not have trivial normal bundles.

To get an idea how Fredholm filtrations are constructed in general, we briefly outline the procedure as follows. Let M be a Fredholm manifold modeled on $\mathcal{E}$. Let $\{E_n\}_n$ be a flag for $\mathcal{E}$ as in Example 3.4(i) above. Choose an index zero Fredholm map $f : M \to \mathcal{E}$ which is transversal to the E_n's, and define $M_n = f^{-1}(E_n)$. Each M_n (when non-empty) is a finite-dimensional submanifold of M of dimension n and $M_n \subset M_{n+1}$. The normal bundle $\nu M_n \to M_n$ of the inclusion $M_n \subset M_{n+1}$ is the pullback $\nu M_n = f^*(\nu E_n)$ of the (trivial) normal bundle $\nu E_n = E_n^{\perp} \cap E_{n+1}$ and, hence, is trivial. The sequence $\{M_n\}_{n=k}^{\infty}$, where $M_k \neq \emptyset$ is the first non-empty submanifold, forms a Fredholm filtration of M. Note that since there is always a *bounded, proper* index zero Fredholm map $f : M \to \mathcal{E}$, the M_n's can be chosen to be *compact*. See the Addendum to Theorem 2C in Eells and Elworthy [19].

One can actually say more about the Fredholm filtrations of a Fredholm manifold, but we need to recall first some facts about the differential geometry of infinite-dimensional manifolds.

Definition 3.5. Let N be a submanifold of M. A *tubular neighborhood of N in M* consist of the following data: a vector bundle $\pi : B \to N$ over N, an open neighborhood V of the zero section $\zeta(N)$ in B, an open set U in M containing N, and a diffeomorphism $f : V \to U$ which commutes with the zero section $\zeta : N \to V$:

$$\begin{tikzcd} V \arrow[d, "\pi|_V"] \arrow[dr, "f"] & \\ N \arrow[u, bend left, "\zeta"] \arrow[r, hook, "i"'] & U \end{tikzcd}$$

U is called the *tube* of the tubular neighborhood. The tubular neighborhood is called *total* if $V = B$ the total space of the bundle.

Using the notion of spray [32, IV.3], its associated exponential map, and restriction to the normal bundle of the inclusion $i : N \to M$, one can prove the existence and uniqueness of tubular neighborhoods, if M is a Hilbert manifold [32, Theorems IV.5.1 and IV.6.2]. On a Riemannian manifold one can always choose tubular neighborhoods to be total.

Definition 3.6. A *Riemannian manifold* is a pair (M, g), where M is a Hilbert manifold, and g is a metric on M, i.e., g_x is a (smoothly varying) positive-definite non-singular symmetric bilinear form on $T_x M$, for every $x \in M$.

According with [32, Corollary II.3.8], every paracompact C^∞-manifold modeled on a separable Hilbert space admits partitions of unity of class C^∞. It follows that Hilbert manifolds admit Riemannian metrics:

Proposition 3.7. [32, Proposition VII.1.1] *Let M be a manifold admitting partitions of unity, and let $\pi : B \to M$ be a vector bundle whose fibers are Hilbertable vector spaces. Then π admits a Riemannian metric.*

Granted all of this, the next statement is a combination of [34, Theorem 2.3] and remarks from [35,19].

Theorem 3.8. *Let M be a Fredholm manifold with Riemannian metric g compatible with the topology of M. There exists a Fredholm filtration $\{M_n\}_{n=k}^\infty$ of M for which geodesically defined exponential neighborhoods Z_n of M_n in M can be constructed satisfying*:

$$Z_n \subset Z_{n+1} \quad \text{and} \quad \bigcup_{n \geqslant k} Z_n = M.$$

Moreover $U_n = Z_n \cap M_{n+1}$ is a tubular neighborhood of M_n in M_{n+1}, for each $n \geqslant k$.

Definition 3.9. We call a Fredholm filtration $\{M_n\}_{n=k}^\infty$ together with a collection $\{U_n\}_{n=k}^\infty$, where U_n is a *total* tubular neighborhood of $M_n \subset M_{n+1}$, an *augmented Fredholm filtration* and we shall denote this by $\mathcal{F} = (M_n, U_n)_{n=k}^\infty$. Note that we assume that each U_n is equipped with a fixed diffeomorphism $\phi_n : \nu M_n \to U_n$, where νM_n is the total space of the normal bundle to the embedding $M_n \hookrightarrow M_{n+1}$.

Fredholm manifolds often arise as spaces of paths and we end this section with one more example. In Section 5, Example 5.10, we will discuss examples of Fredholm manifolds arising from loop groups ΩG of certain compact Lie groups G (and their associated spin structures.)

Example 3.10. (See [20].) Let X be a complete finite-dimensional Riemannian manifold, and $a \in X$. Let $M = P_a(X)$ be the space of paths $\gamma : [0, 1] \to X$, with $\gamma(0) = a$ and γ absolutely continuous with square integrable derivative. Then M is a separable smooth Hilbert manifold. Moreover a complete Riemannian structure on M is given by

$$g_\gamma(u, v) = \langle u, v\rangle_\gamma = \int_0^1 \langle D_\gamma u, D_\gamma v\rangle_\gamma,$$

for $u, v \in T_\gamma M$, where D_γ denotes the covariant derivative along γ. There is natural diffeomorphism

$$\delta : P_a(X) \to P_0(T_a X), \quad \delta(\gamma)(t) = \int_0^t \tau_0^s \gamma'(s)\,\mathrm{d}s,$$

where τ_0^s denotes parallel transport along γ from $T_{\gamma(s)}X$ to $T_a X$. This map δ, called E. Cartan's development map, gives a diffeomorphism of $M = P_a(X)$ with the Hilbert space $P_0(T_a X)$ and, hence, a unique Fredholm structure on the contractible space M.

4. The C^*-algebra of a Fredholm manifold

Let M be a smooth, separable, connected, paracompact Hilbert manifold modeled on the separable, infinite-dimensional Euclidean space $\mathcal{E}$. We assume that M is equipped with a Riemannian–Fredholm structure, i.e., a reduction of the structure group of M from $GL(\mathcal{E})$ to $GL_{\mathcal{K}}(\mathcal{E})$ and a Riemannian metric g that is compatible with the topology of M. This is equivalent to a reduction of the structure group from $GL(\mathcal{E})$ to $\mathcal{O}_{\mathcal{K}}(\mathcal{E}) = GL_{\mathcal{K}}(\mathcal{E}) \cap O(\mathcal{E})$. (See Section 5.2.)

Let $\mathcal{F} = (M_n, U_n)_{n=k}^{\infty}$ be an augmented Fredholm filtration of M by closed n-dimensional submanifolds M_n with total tubular neighborhoods $M_n \subset U_n \subset M_{n+1}$, as in Definition 3.9. Let $p_n : \nu M_n \to M_n$ denote the normal bundle of the embedding $j_n : M_n \hookrightarrow M_{n+1}$. That is, we have a short exact sequence

$$0 \to TM_n \to TM_{n+1}|_{M_n} \to \nu M_n \to 0$$

of finite rank vector bundles.

These geometric considerations lead us to the following topological diagram of bundles and spaces:

$$\begin{tikzcd} \nu M_n \arrow[r, "\phi_n", "\text{diffeo}"'] \arrow[d, "p_n"', "\text{normal}"] & U_n \arrow[r, hook, "k_n", "\text{open}"'] & M_{n+1} \\ M_n & & \end{tikzcd} \tag{5}$$

where the tubular neighborhood U_n is identified with the total space of the normal bundle νM_n via a fixed diffeomorphism $\phi_n : \nu M_n \to U_n$ and $k_n : U_n \hookrightarrow M_{n+1}$ denotes the (open) inclusion.

For each n, let M_n have the induced Riemannian metric $g_n = i_n^*(g)$ where $i_n : M_n \hookrightarrow M$ denotes the inclusion. Thus, for each $n \geqslant k$, we have the associated C^*-algebra

$$\mathcal{A}(M_n) = \mathcal{SC}(M_n) = \mathcal{S} \mathbin{\widehat{\otimes}} C_0\big(M_n, \mathrm{Cliff}(TM_n)\big)$$

as in Definition 2.3. Recall that $\mathcal{S}$ denotes the C^*-algebra $C_0(\mathbb{R})$ graded by even and odd functions.

The restricted bundle $TM_{n+1}|_{M_n}$ is the pullback bundle $j_n^*(TM_{n+1})$ under the inclusion $j_n : M_n \hookrightarrow M_{n+1}$. Thus, there is an induced pullback metric $j_n^*(g)$ and pullback connection $j_n^*(\nabla^{n+1})$ on $TM_{n+1}|_{M_n}$, where ∇^{n+1} is the Levi-Civita connection of M_{n+1} [5]. Using this pullback metric we have an orthogonal splitting

$$TM_{n+1}|_{M_n} \cong TM_n \oplus \nu M_n$$

of vector bundles on M_n. Give νM_n the induced bundle metric and projected connection $\nabla^{\nu M_n}$. Thus, $p_n : \nu M_n \to M_n$ has a canonical structure as an affine Euclidean bundle. By Theorem 2.11, there is an induced C^*-algebra homomorphism

$$\Psi_{p_n} : \mathcal{A}(M_n) \to \mathcal{A}(\nu M_n)$$

where νM_n is given the Riemannian metric from Lemma 2.6.

Give the open set $U_n \subset M_{n+1}$ the induced Riemannian metric $k_n^*(g_{n+1})$ from M_{n+1}. By Lemma 2.5 we have an inclusion of C^*-algebras

$$(k_n)_* : \mathcal{A}(U_n) \hookrightarrow \mathcal{A}(M_{n+1})$$

induced by the inclusion $k_n : U_n \hookrightarrow M_{n+1}$. Finally, we have by Lemma 2.4, a canonical C^*-algebra isomorphism

$$(\phi_n)_* : \mathcal{A}(\nu M_n) \xrightarrow{\cong} \mathcal{A}(U_n)$$

induced by the diffeomorphism $\phi_n : \nu M_n \to U_n$ of the tubular neighborhood U_n with the total space νM_n of the normal bundle.

Thus, we have the following diagram of C^*-algebras and $*$-homomorphisms, which can be considered as the non-commutative version of diagram (5) above:

$$\begin{tikzcd} \mathcal{A}(\nu M_n) \arrow[r, "(\phi_n)_*", "\cong"'] & \mathcal{A}(U_n) \arrow[r, hook, "(k_n)_*"] & \mathcal{A}(M_{n+1}) \\ \mathcal{A}(M_n) \arrow[u, "\Psi_{p_n}", "\text{Thom}"'] \arrow[urr, dashed, "\alpha_n"'] & & \end{tikzcd} \tag{6}$$

The dotted arrow, which is by definition the composition of the other three, gives the connecting map $\alpha_n : \mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$ in the definition of our C^*-algebra $\mathcal{A}(M, g, \mathcal{F})$.

Definition 4.1. Let M be a smooth Fredholm manifold,[4] modeled on the separable infinite-dimensional Euclidean space $\mathcal{E}$, equipped with a Riemannian metric g compatible with the topology of M, and an augmented Fredholm filtration $\mathcal{F}=(M_n,U_n)_{n=k}^{\infty}$. The *$C^*$-algebra of the triple* $(M,g,\mathcal{F})$ is the direct limit C^*-algebra

$$\mathcal{A}(M,g,\mathcal{F})=\varinjlim \mathcal{A}(M_n) \tag{7}$$

where the direct limit is taken over the directed system $\{\mathcal{A}(M_n),\alpha_n\}_{n=k}^{\infty}$ and the connecting maps α_n are given by diagram (6).

It easily follows that $\mathcal{A}(M,g,\mathcal{F})$ has the structure of a $\mathbb{Z}_2$-graded, separable, nuclear C^*-algebra. One can also show (using Lemma 2.4 and the construction in Lemma 2.6) that $\mathcal{A}(M,g,\mathcal{F})$ does not depend, up to isomorphism of $\mathbb{Z}_2$-graded C^*-algebras, on the choice of the Riemannian metric g of M. Indeed, we have:

Lemma 4.2. *Let M be a smooth Fredholm manifold with augmented Fredholm filtration $\mathcal{F}=(M_n,U_n)_{n=k}^{\infty}$. If g and h are Riemannian metrics on M compatible with the topology, there is a canonical map*

$$\Phi:\mathcal{A}(M,g,\mathcal{F})\to\mathcal{A}(M,h,\mathcal{F})$$

which is an isomorphism of $\mathbb{Z}_2$-graded C^*-algebras.

Proof. The identity map $\mathrm{id}_M:(M,g)\to(M,h)$ is a diffeomorphism of Riemannian Fredholm manifolds and induces for each $n\geqslant k$ a commuting diagram

$$\begin{array}{ccccccc}
\mathcal{A}(M_n,g_n) & \longrightarrow & \mathcal{A}(\nu M_n,g_n') & \longrightarrow & \mathcal{A}(U_n,k_n^*(g_{n+1})) & \longrightarrow & \mathcal{A}(M_{n+1},g_{n+1}) \\
\big\downarrow\cong & & \big\downarrow\cong & & \big\downarrow\cong & & \big\downarrow\cong \\
\mathcal{A}(M_n,h_n) & \longrightarrow & \mathcal{A}(\nu M_n,h_n') & \longrightarrow & \mathcal{A}(U_n,k_n^*(h_{n+1})) & \longrightarrow & \mathcal{A}(M_{n+1},h_{n+1})
\end{array}$$

where g_n' and h_n' are the Riemannian metrics induced on the total space νM_n by Lemma 2.6 and the vertical maps are the $\mathbb{Z}_2$-graded C^*-algebra isomorphisms induced by $\mathrm{id}_{M_n}:(M_n,g_n)\to(M_n,h_n)$ from Lemma 2.4. The result now easily follows by the universal property for direct limits [46] since the composition of the top and bottom rows are the connecting maps in the direct limits $\mathcal{A}(M,g,\mathcal{F})$ and $\mathcal{A}(M,h,\mathcal{F})$, respectively. □

The C^*-algebra $\mathcal{A}(M,g,\mathcal{F})$ does indeed depend on the choice of the augmented Fredholm filtration $\mathcal{F}=(M_n,U_n)_{n=k}^{\infty}$. However, we will see in the next section that the K-theory groups of $\mathcal{A}(M,g,\mathcal{F})$ do not depend on the choice of the tubular neighborhoods $\{U_n\}_{n=k}^{\infty}$ and, moreover, if M has an appropriate spin structure then the K-theory groups do not depend on the choice of filtrating manifolds $\{M_n\}_{n=k}^{\infty}$.

We will now consider two examples from the literature that are directly related to this construction.

Example 4.3. Consider $M=\mathcal{E}$, with metric g induced by the inner product $\langle\cdot,\cdot\rangle$, and Fredholm filtration given by a flag $\{E_n\}_n$ of $\mathcal{E}$ as in Example 3.4(i). Setting $\nu E_n=U_n=E_{n+1}$, we obtain an augmented Fredholm filtration $\mathcal{F}=(E_n,E_{n+1})_{n=1}^{\infty}$ of $\mathcal{E}$. We thus have the C^*-algebra

$$\mathcal{A}(\mathcal{E},g,\mathcal{F})=\varinjlim \mathcal{A}(E_n)$$

as constructed above. Since $TE_n\cong E_n\times E_n$ is trivial, we have that

$$\mathcal{A}(E_n)\cong\mathcal{S}\mathbin{\widehat{\otimes}}C_0\big(E_n,\mathrm{Cliff}(E_n)\big)=\mathcal{SC}(E_n)$$

as in Definition 3.1 of Higson–Kasparov–Trout [25]. Also, since $\nu E_n=U_n=E_{n+1}$, it follows that the connecting map $\alpha_n:\mathcal{A}(E_n)\to\mathcal{A}(E_{n+1})$ can be canonically identified with the Bott periodicity map

$$\beta_{(n+1)n}=\alpha_n:\mathcal{A}(E_n)\to\mathcal{A}(E_{n+1})$$

[4] Recall that we assume M to be connected, separable, paracompact, and Hausdorff.

of [25, Definition 3.1]. Using an approximation argument to deal with the dense subalgebra of compactly supported functions, it follows that the C^*-algebra $\mathcal{A}(\mathcal{E}, g, \mathcal{F})$ is isomorphic to the C^*-algebra

$$\mathcal{A}(\mathcal{E}) = \varinjlim_{E_a \subset \mathcal{E}} \mathcal{A}(E_a)$$

where the direct limit is taken over the directed system of *all* finite-dimensional subspaces $E_a \subset \mathcal{E}$. See also Lemma 2.6 and the discussion after Definition 4.6 of Higson–Kasparov [24]. This C^*-algebra has important applications to the Baum–Connes and Novikov Conjectures [24,25,49]

Example 4.4. Another example, which generalizes the previous one, comes from the Thom isomorphism theorem for infinite rank Euclidean vector bundles [45]. Suppose M is the total space of a smooth (locally trivial) vector bundle $p : M \to X$, with fiber $\mathcal{E}$ and structure group $GL(\mathcal{E})$, over a smooth, *finite-dimensional* Riemannian manifold X of dimension k. Since the fiber $\mathcal{E}$ is infinite-dimensional, we may assume [16] that $M = X \times \mathcal{E}$ is trivial. The inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$ then canonically induces a Euclidean metric structure on the bundle M. Using the isomorphism

$$TM \cong TX \times T\mathcal{E} = TX \times (\mathcal{E} \times \mathcal{E})$$

we canonically endow the total space M with the structure of a Riemannian–Hilbert manifold. Also, since TM is trivial, it follows that M has a canonical structure as a Fredholm manifold.

Let $\{E_n\}_{n=1}^{\infty}$ be a flag for $\mathcal{E}$. For each $n \geqslant k + 1$, let

$$M_n = X \times E_{n-k} \to X$$

denote the trivial vector subbundle of rank $n - k$. One can then check that the collection of submanifolds $\{M_n\}_{n=k+1}^{\infty}$ determines a Fredholm filtration of M such that we can canonically identify the total space νM_n of the normal bundle of $M_n \hookrightarrow M_{n+1}$ as M_{n+1}. We then have that $\mathcal{F} = (M_n, M_{n+1})_{n=k+1}^{\infty}$ is an augmented Fredholm filtration for M. Since $M_n = X \times E_{n-k}$ we have

$$\mathcal{A}(M_n) \cong \mathcal{A}(E_{n-k}) \mathbin{\widehat{\otimes}} \mathcal{C}(X) \cong \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{C}(E_{n-k}) \mathbin{\widehat{\otimes}} \mathcal{C}(X).$$

It follows from Proposition 2.13, the results in [45], and a similar approximation argument that

$$\mathcal{A}(M, g, \mathcal{F}) \cong \mathcal{A}(\mathcal{E}) \mathbin{\widehat{\otimes}} \mathcal{C}(X) \cong \mathcal{A}(M, \nabla_0, X)$$

where $\mathcal{A}(M, \nabla_0, X)$ is the C^*-algebra of the affine Euclidean bundle $p : M \to X$, equipped with the trivial connection $\nabla_0 = d$, as in [45, Definition 3.11].

5. *K*-theory, spin structures and Poincaré duality

In this section we discuss the relationship between the topological K-theory groups, the (compactly supported) K-homology groups of a Fredholm manifold M and the K-theory groups of the C^*-algebra $\mathcal{A}(M, g, \mathcal{F})$ we constructed in the last section. When an oriented Riemannian–Fredholm manifold M has been equipped with an appropriate infinite-dimensional spin structure, we will see that all of these groups coincide, as in the finite-dimensional spin manifold setting.

5.1. The topological K-theory of a Fredholm manifold

Mukherjea [35, Section 2], in the context of generalized cohomologies obtained from a spectrum on the category of compact spaces, defined the corresponding cohomology groups for Fredholm manifolds. Based on his work, we are led to make the following definition.

Definition 5.1. Let M be smooth Fredholm manifold with augmented Fredholm filtration $\mathcal{F} = (M_n, U_n)_{n=k}^{\infty}$. The *jth topological K-theory group of* $(M, \mathcal{F})$, denoted $K^{\infty - j}(M, \mathcal{F})$, is defined to be the direct limit

$$K^{\infty - j}(M, \mathcal{F}) = \varinjlim K^{n-j}(M_n), \quad \text{for } j = 0, 1,$$

where the connecting maps are the Gysin (or shriek) maps [12,27]

$$(j_n)_! : K^{n-j}(M_n) \to K^{n+1-j}(M_{n+1})$$

associated to the inclusions $j_n : M_n \hookrightarrow M_{n+1}$. These may be obtained from diagram (5), via the functoriality properties of topological K-theory, as the composition of Gysin maps

$$\begin{tikzcd} K^{n+1-j}(\nu M_n) \arrow[r, "(\phi_n)_!", "\cong"'] & K^{n+1-j}(U_n) \arrow[r, "(k_n)_!"] & K^{n+1-j}(M_{n+1}) \\ K^{n-j}(M_n) \arrow[u, "s_!=\mathrm{Thom}", "\cong"'] \arrow[urr, dashed, "(j_n)_!"'] & & \end{tikzcd} \tag{8}$$

where the map $s_!$ is the Gysin map associated to the zero section $s : M_n \to \nu M_n$, and which induces the Thom isomorphism. (Compare this with diagram (6).)

Clearly, the definition of the topological K-theory of M does not depend on the choice of tubular neighborhoods $\{U_n\}_n$ (or any Riemannian metric g) but does, *a priori*, depend on the choice of Fredholm filtration $\{M_n\}_n$, as does the definition of $\mathcal{A}(M, g, \mathcal{F})$. However, if M has a certain infinite-dimensional spin structure, then these topological K-theory groups $K^{\infty-j}(M, \mathcal{F})$ do not depend on the choice of $\mathcal{F} = (M_n, U_n)_n$.

5.2. *Fredholm* Spin_q*-structures*

Recall the notation introduced at the beginning of Section 3. Let $\mathcal{E}$ be a separable infinite-dimensional Euclidean space. For $1 \leqslant q \leqslant \infty$, let $GL_q(\mathcal{E}) = GL(\mathcal{E}) \cap (I + P_q)$, where P_q is the qth Schatten–von Neumann perturbation class. Let $\mathcal{O}(\mathcal{E})$ denote the orthogonal operators on $\mathcal{E}$. We let $\mathcal{O}_q(\mathcal{E}) = \mathcal{O}(\mathcal{E}) \cap GL_q(\mathcal{E})$ and let $\mathcal{SO}_q(\mathcal{E})$ denote the connected component of I in $\mathcal{O}_q(\mathcal{E})$. All of these groups are infinite-dimensional Banach–Lie groups [13] with manifold topology given by the restriction of the norm $\|\cdot\|_q$. Note that since $P_q \subset \mathcal{K}(\mathcal{E})$, it follows that $GL_q(\mathcal{E}) \subset GL_{\mathcal{K}}(\mathcal{E})$ and so any Hilbert manifold with $GL_q(\mathcal{E})$ as structure group has a canonical Fredholm structure as in Definition 3.2.

Let M be a smooth, paracompact, connected Hilbert manifold, without boundary, modeled on $\mathcal{E}$. Let $\xi : E \to M$ be a smooth (locally trivial) vector bundle over M, with fiber $\mathcal{E}$, endowed with a reduction of the structure group from $GL(\mathcal{E})$ to $GL_q(\mathcal{E})$. A *Riemannian q-structure* [2, Definition 2.1] on ξ is a reduction of the structure group from $GL_q(\mathcal{E})$ to $\mathcal{O}_q(\mathcal{E})$. Since M is paracompact, this may be accomplished by using a partition of unity to define a smooth bundle metric g_x on the fibers E_x of ξ. If ξ is the tangent bundle $\pi : TM \to M$, with Fredholm structure group $GL_q(\mathcal{E})$, then we say that M has a Riemannian q-structure.

Definition 5.2. [2, Definition 2.2] A Riemannian q-structure on $\xi : E \to M$ is *orientable* if ξ admits a further reduction of its structure group to $\mathcal{SO}_q(\mathcal{E})$. A given reduction will be called an *orientation* and ξ will be said to have an *oriented Riemannian q-structure*.

A proof of the following can be found in [30, Proposition 6.2] or [2, Theorem 2.1].

Theorem 5.3. *A Riemannian q-structure on $\xi : E \to M$ is orientable if and only if the first Stieffel–Whitney class $w_1(\xi) \in H^1(M, \mathbb{Z}_2)$ vanishes. In particular, if M has a Riemannian q-structure, then M is orientable if and only if $w_1(M) = w_1(TM) = 0$.*

For the theory of Stieffel–Whitney classes associated to Hilbert bundles over Hilbert manifolds that we are considering, see Koschorke [30]. Note that, contrary to the finite-dimensional case, these characteristic classes are *not* diffeomorphism invariants, in general. (See [30, Example 6.2] for details.)

Since $\mathcal{SO}_q(\mathcal{E})$ is of index 2 in $\mathcal{O}_q(\mathcal{E})$, it follows that the universal covering $\mathrm{Spin}_q(\mathcal{E})$ is a Banach–Lie group and the covering map is 2-sheeted. We thus have an exact sequence of (paracompact) topological groups

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_q(\mathcal{E}) \xrightarrow{\rho} \mathcal{SO}_q(\mathcal{E}) \to 1.$$

Concrete realizations of these infinite-dimensional spin groups were constructed for $q = 1$ by P. de la Harpe [14] and for $q = 2$ by Plymen and Streater [40]. However, we will not need explicit constructions of these spin groups, only the fact that they are 2-sheeted covering groups of the associated special orthogonal groups, as in the finite-dimensional case. In the following, we may abbreviate Spin_q and $\mathcal{SO}_q$ for $\mathrm{Spin}_q(\mathcal{E})$ and $\mathcal{SO}_q(\mathcal{E})$, respectively.

Definition 5.4. [2, Definition 2.4] Suppose $\xi : E \to M$ has an $\mathcal{SO}_q$-structure, i.e., an oriented Riemannian q-structure. A Spin_q*-structure* on ξ is a principal bundle extension associated to the covering map

$$\rho : \mathrm{Spin}_q \to \mathcal{SO}_q$$

of the principal $\mathcal{SO}_q$-bundle of linear frames of ξ. If M is a Fredholm manifold with oriented Riemannian q-structure, then a Spin_q*-structure* on M is a Spin_q-structure on $\pi : TM \to M$. We will then call M a *Fredholm* Spin_q*-manifold*.

That is, if $p : L \to M$ is the principal $\mathcal{SO}_q$-bundle of oriented orthonormal frames of $\xi : E \to M$, then a Spin_q-structure for ξ is a principal Spin_q-bundle $q : \Sigma \to M$ such that Σ is a 2-fold covering of L, the restriction of the covering map $\tilde{\rho} : \Sigma \to L$ to the fibers are 2-sheeted coverings and

$$\tilde{\rho}(s \cdot g) = \tilde{\rho}(s)\rho(g) \quad \text{and} \quad q(s) = p\big(\tilde{\rho}(s)\big)$$

for all $s \in \Sigma$ and $g \in \mathrm{Spin}_q$. Thus, the following diagram commutes:

$$\begin{tikzcd} \Sigma \arrow[r, "\tilde{\rho}"] \arrow[d, "q"'] & L \arrow[d, "p"] \\ M \arrow[r, "\mathrm{Id}_M"] & M \end{tikzcd}$$

For $q = 1$ de la Harpe has shown that the existence of a Spin_q-structure on a Fredholm manifold M with oriented Riemannian q-structure is equivalent to the vanishing $w_2(M) = 0$ of the second Stieffel–Whitney class in $H^2(M, \mathbb{Z}_2)$. We wish to extend his result to all values $1 \leqslant q \leqslant \infty$ and all $\mathcal{SO}_q$-vector bundles. Although his argument for $q = 1$ almost certainly holds in the general case, we will provide a more direct proof using an argument of Lawson and Michelson [33] from the finite-dimensional spin case. In order to do that, we need the following cohomology computation, which follows from some results in the literature [13,15], but we provide a proof for completeness.

Lemma 5.5. *For* $1 \leqslant q \leqslant \infty$, $H^1(\mathcal{SO}_q(\mathcal{E}), \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Proof. Choose a flag $\{E_n\}$ for $\mathcal{E}$ as in Example 3.4(i). This induces an inclusion of topological groups

$$SO(\infty) = \varinjlim SO(n) \hookrightarrow \mathcal{SO}_q(\mathcal{E})$$

which, by Proposition 3 in [15], is a homotopy equivalence. Hence, using the identity as basepoint, we have by Bott periodicity [7]:

$$\pi_1\big(\mathcal{SO}_q(\mathcal{E})\big) \cong \pi_1\big(SO(\infty)\big) \cong \varinjlim \pi_1\big(SO(n)\big) \cong \mathbb{Z}_2.$$

Since $\mathcal{SO}_q(\mathcal{E})$ is connected with Abelian fundamental group, it follows that

$$H_1\big(\mathcal{SO}_q(\mathcal{E})\big) \cong \pi_1\big(\mathcal{SO}_q(\mathcal{E})\big) \cong \mathbb{Z}_2.$$

The result now follows from the Universal Coefficient Theorem in cohomology:

$$H^1\big(\mathcal{SO}_q(\mathcal{E}), \mathbb{Z}_2\big) \cong \mathrm{Hom}\big(H_1\big(\mathcal{SO}_q(\mathcal{E}), \mathbb{Z}\big), \mathbb{Z}_2\big) \cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$$

and we are done. □

Theorem 5.6. *Let* $\xi : E \to M$ *be a Hilbert bundle with oriented Riemannian* q*-structure. Then* ξ *has a* Spin_q*-structure if and only if the second Stieffel–Whitney class* $w_2(\xi) \in H^2(M, \mathbb{Z}_2)$ *vanishes. In particular, if* M *is a Fredholm manifold with oriented Riemannian* q*-structure, then there exists a* Spin_q*-structure on* M *if and only if* $w_2(M) = 0$.

For the following, recall that in principal bundle theory, if M is a paracompact space and G is a topological group, then $H^1(M, G)$ is isomorphic to the set of isomorphism classes of principal G-bundles on M, where we are using Čech cohomology. (See Appendix A of Lawson and Michelsohn [33].)

Proof. Let $p : L \to M$ be the principal $\mathcal{SO}_q$-bundle of oriented orthonormal frames of ξ. We then have a fibration

$$\mathcal{SO}_q(\mathcal{E}) \xrightarrow{i} L \xrightarrow{p} M$$

which induces an exact sequence

$$H^1(M,\mathbb{Z}_2) \xrightarrow{p^*} H^1(L,\mathbb{Z}_2) \xrightarrow{i^*} H^1\big(\mathcal{SO}_q(\mathcal{E}),\mathbb{Z}_2\big) \xrightarrow{\delta_\xi} H^2(M,\mathbb{Z}_2)$$

in Čech cohomology. It follows by the above discussion (see also [2, Theorem 2.3]) that ξ has a Spin_q-structure if and only if there is a cohomology class $\alpha=\alpha(\xi)\in H^1(L,\mathbb{Z}_2)$ such that $i^*(\alpha)\neq 0$ since a Spin_q-structure on ξ determines a non-trivial 2-sheeted covering of L. Let g_2 be the generator of $H^1(\mathcal{SO}_q(\mathcal{E}),\mathbb{Z}_2)\cong\mathbb{Z}_2$. It follows that ξ has a Spin_q-structure if and only if there is a cohomology class $\alpha=\alpha(\xi)\in H^1(L,\mathbb{Z}_2)$ such that $i^*(\alpha)=g_2$. Consequently, by exactness of the sequence above, we have that this holds if and only if

$$w_2(\xi)=\delta_\xi(g_2)=\delta_\xi\big(i^*(\alpha)\big)=0\in H^2(M,\mathbb{Z}_2).$$

The fact that the second Stieffel–Whitney class of ξ is given by

$$w_2(\xi)=\delta_\xi(g_2)\in H^2(M,\mathbb{Z}_2)$$

follows from the universal properties of these classes [30, Proposition 6.3]. □

Consequently, if $\xi:E\to M$ admits a Spin_q-structure determined by $\alpha(\xi)\in H^1(L,\mathbb{Z}_2)$ then the most general Spin_q-structure on ξ is of the form $\alpha(\xi)+p^*(\beta)$ where $\beta\in H^1(M,\mathbb{Z}_2)$. Thus, there is a bijection between the set of (isomorphism classes of) Spin_q-structures on ξ and $H^1(M,\mathbb{Z}_2)$. It follows that a Spin_q-structure on ξ (or M) is unique if $H^1(M,\mathbb{Z}_2)=0$.

The next two results are immediate corollaries (see [2, Theorems 2.5 and 2.6].)

Proposition 5.7. *Given* Spin_q*-structures on two out of the three vector bundles ξ_1, ξ_2, and $\xi_1\oplus\xi_2$ on M, there is a uniquely determined* Spin_q*-structure on the third.*

Proposition 5.8. *If $\xi:E\to M$ admits a* Spin_q*-structure and $f:N\to M$ is smooth, then the pull-back vector bundle $f^*\xi:f^*E\to N$ admits a* Spin_q*-structure.*

In the context of Fredholm manifolds, the above give:

Corollary 5.9. *Let M be a Fredholm* Spin_q*-manifold. If $\{M_n\}_n$ is any associated Fredholm filtration of M then each M_n has a canonical* (*finite-dimensional*) *spin structure.*

Indeed, associated to the inclusion $i_n:M_n\to M$ we have a split short exact sequence

$$0\to TM_n\to TM|_{M_n}\to \nu M_n\to 0.$$

The normal bundle νM_n has a Spin_q-structure being trivial, and $TM|_{M_n}=i_n^*(TM)$ has one because of Proposition 5.8. Thus, we have

$$w_2(\nu M_n)=w_2(TM|_{M_n})=0$$

and finally Proposition 5.7 gives the result since $w_2(M_n)=0$.

We end this subsection about spin structures with an example coming from certain based loop groups.

Example 5.10. Consider a compact, connected, simply connected, simple Lie group G. Let $\Omega_sG=H^s_0(S^1,G)$ be the *group of based loops on G*, i.e., maps from the circle to G in the sth Sobolev space H^s which take a fixed point on S^1 into the identity element of G, where $s\geqslant 1/2$. Ω_sG is a (real) Hilbert–Lie group.

D. Freed constructed in [22, Section 5] a particular Fredholm 1-structure, coming from a classifying map

$$\Omega_sG\to BGL(\infty;\mathbb{C})\sim\Phi_0,$$

where Φ_0 denotes the Fredholm operators of index zero. The resulting frame bundle was called the *geometric frame bundle*. He concluded that the realization of this geometric frame bundle is trivial and that the Stieffel–Whitney classes of $\Omega_s G$ vanish [22, Theorem 5.30]. Our Theorem 5.6 now shows that this is the unique Spin_1-structure on $\Omega_s G$. Indeed, the hypothesis on G implies that $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$, and $\pi_3(G) = \mathbb{Z}$. Consequently $H_1(\Omega_s G, \mathbb{Z}) = 0$ and $H_2(\Omega_s G, \mathbb{Z}) \cong \pi_2(\Omega_s G) \cong \pi_3(G) = \mathbb{Z}$. These imply that $H^1(\Omega_s G, \mathbb{Z}_2) = 0$ and $H^2(\Omega_s G, \mathbb{Z}_2) = \mathbb{Z}/2$. As $w_2(\Omega_s G) = 0$ by Freed's Corollary 5.31, and as Spin_1-structures on ΩG are parametrized by $H^1(\Omega_s G, \mathbb{Z}_2) = 0$, we obtain the claimed uniqueness of the Spin_1-structure on $\Omega_s G$. Moreover, Freed's–Fredholm structure is actually the unique Spin_q-structure, for all $1 \leqslant q \leqslant \infty$.

5.3. K-homology and Poincaré duality

Recall that if X is a compact space then the jth K-homology group of X is the Abelian group $K_j(X) = KK^j(C(X), \mathbb{C})$ which is dual to the jth K-theory group $K^j(X) \cong KK^j(\mathbb{C}, C(X))$. The map $X \mapsto K_j(X)$ defines a generalized homology theory on the category of compact spaces and continuous maps [8,28,26].

Definition 5.11. Let M be a paracompact space. The *jth compactly supported K-homology group of M* is

$$K_j^c(M) = \varinjlim_{X \subset M} K_j(X),$$

where the direct limit is over all the *compact* subsets $X \subset M$, and $j = 0, 1$.

In order to prove our Poincaré duality result, we need the following result, whose proof requires the KK-theory for pro-C^*-algebras developed by Weidner [47] and Phillips [38]. A heuristic proof would be that since $M \sim M_\infty = \varinjlim M_n$, we have in compactly supported K-homology that $K_j^c(M) \cong K_j^c(M_\infty) \cong \varinjlim K_j^c(M_n)$.

Proposition 5.12. *Let M be a smooth Fredholm manifold. If $\{M_n\}_{n=k}^\infty$ is any Fredholm filtration of M then there is an isomorphism of Abelian groups*

$$K_j^c(M) \cong \varinjlim K_j^c(M_n), \quad j = 0, 1,$$

where the connecting map $K_j^c(M_n) \to K_j^c(M_{n+1})$ in the direct limit is induced by the inclusion $M_n \hookrightarrow M_{n+1}$.

Proof. Let g be a Riemannian metric on M compatible with the topology (which exists via paracompactness). Thus, (M, g) is a metric space. Since metric spaces are compactly generated [48, I.4.3], it follows that the algebra $C(M)$ of all continuous complex-valued functions on M, with the topology of uniform convergence on compact subsets, is a pro-C^*-algebra with involution given by pointwise complex conjugation [37, Exercise 1.3.3]. Let $\mathcal{C}_M$ denote the collection of all compact subsets X of M ordered by inclusion. Since M is regular, it is completely Hausdorff [37, Definition 2.2], and so by [37, Corollary 2.9], it follows that there is an isomorphism

$$C(M) \cong \varprojlim_{X \in \mathcal{C}_M} C(X) \tag{9}$$

of pro-C^*-algebras. Similarly, for each n, we have an isomorphism

$$C(M_n) \cong \varprojlim_{K_n \in \mathcal{C}_{M_n}} C(K_n) \tag{10}$$

of pro-C^*-algebras where $\mathcal{C}_{M_n}$ denotes the set of all compact subsets K_n of M_n ordered by inclusion. Let $M_\infty = \bigcup_n M_n = \varinjlim M_n$ with the direct limit topology. Since M_∞ is countably compactly generated in the direct limit topology, we then have an isomorphism

$$C(M_\infty) \cong \varprojlim_n C(M_n) \tag{11}$$

of pro-C^*-algebras. By Theorem 3.3 the inclusion $M_\infty \hookrightarrow M$ is a homotopy equivalence, hence the pro-C^*-algebras $C(M)$ and $C(M_\infty)$ have the same homotopy type.

Using the fact that Weidner's KK-groups $KK_W^j(A,B)$ for pro-C^*-algebras [47,38] extend Kasparov's KK-groups for C^*-algebras [28], are homotopy-invariant, and convert inverse limits to direct limits[5] in the K-homology variable, we compute as follows:

$$\begin{aligned}
K_j^c(M) &= \varinjlim_{X\in\mathcal{C}_M} KK^j\big(C(X),\mathbb{C}\big) && \text{(Definition 5.11)}\\
&\cong KK_W^j\Big(\varprojlim_{X\in\mathcal{C}_X} C(X),\mathbb{C}\Big) && \text{(by [47, Theorem 5.1])}\\
&\cong KK_W^j\big(C(M),\mathbb{C}\big) && \text{(by Eq. (9))}\\
&\cong KK_W^j\big(C(M_\infty),\mathbb{C}\big) && \text{(homotopy invariance)}\\
&\cong KK_W^j\Big(\varprojlim_n C(M_n),\mathbb{C}\Big) && \text{(by Eq. (11))}\\
&\cong \varinjlim_n KK_W^j\big(C(M_n),\mathbb{C}\big) && \text{(by [47, Theorem 5.1])}\\
&\cong \varinjlim_n KK_W^j\Big(\varprojlim_{K_n} C(K_n),\mathbb{C}\Big) && \text{(by Eq. (10))}\\
&\cong \varinjlim_n \varinjlim_{K_n} KK^j\big(C(K_n),\mathbb{C}\big) && \text{(by [47, Theorem 5.1])}\\
&\cong \varinjlim_n K_j^c(M_n) && \text{(Definition 5.11).} \qquad \square
\end{aligned}$$

Compare the following result for Fredholm Spin_q-manifolds with [35, Theorem 2.1].

Theorem 5.13 *(Poincaré duality). If M is a smooth Fredholm Spin_q-manifold with augmented Fredholm filtration $\mathcal{F}$, there is an isomorphism*

$$K^{\infty-j}(M,\mathcal{F}) \cong K_j^c(M).$$

Proof. Let $\mathcal{F}=(M_n,U_n)_{n=k}^\infty$ be the augmented Fredholm filtration. Since M is a Fredholm Spin_q-manifold, each M_n has a canonical spin structure by Corollary 5.9. By [43, Corollary 31] (or [26, Exercise 11.8.11]) we have a natural Poincaré duality isomorphism

$$P_n : K^{n-j}(M_n) \xrightarrow{\cong} K_j^c(M_n)$$

given by the cap product with the fundamental class $[M_n]$. Naturality is the assertion that the Poincaré duality diagram

$$\begin{tikzcd}
K^{n-j}(M_n) \arrow[r, "j_{n!}"] & K^{n+1-j}(M_{n+1}) \\
K_j^c(M_n) \arrow[u, "P_n", "\cong"'] \arrow[r, "j_{n*}"] & K_j^c(M_{n+1}) \arrow[u, "\cong", "P_{n+1}"']
\end{tikzcd}$$

commutes, where $j_n : M_n \hookrightarrow M_{n+1}$. It now follows that:

$$\begin{aligned}
K_j^c(M) &\cong \varinjlim K_j^c(M_n) && \text{(Proposition 5.12)}\\
&\cong \varinjlim K^{n-j}(M_n) && \text{(classical Poincaré duality)}\\
&= K^{\infty-j}(M,\mathcal{F}) && \text{(Definition 5.1)}
\end{aligned}$$

as desired. $\square$

[5] Note that there is a typo in the statement of [47, Theorem 5.1].

5.4. K-theory of the C^*-algebra $\mathcal{A}(M, g, \mathcal{F})$

First we discuss the finite-dimensional results we will need. Let M_n be an oriented Riemannian n-manifold. An important relationship between the non-commutative C^*-algebra $\mathcal{C}(M_n) = C_0(M_n, \mathrm{Cliff}(TM_n))$ and the commutative C^*-algebra $C_0(M_n)$ is given by spin^c-structures [33]. Let $\mathbb{C}_1 = \mathrm{Cliff}(\mathbb{R})$ denote the first complex Clifford algebra. The following is adapted from Theorem 2.11 of Plymen [39] and Proposition II.A.9 of Connes [11].

Proposition 5.14. *If $n = 2k$ is even, there is a bijective correspondence between* spin^c*-structures on M_n and Morita equivalences* (*in the sense of Rieffel* [42,44]) *between the C^*-algebras $C_0(M_n)$ and $\mathcal{C}(M_n)$. Thus, $\mathcal{A}(M_n)$ is Morita equivalent to $C_0(\mathbb{R} \times M_n)$. If $n = 2k + 1$ is odd, then* spin^c*-structures on M are in bijective correspondence with Morita equivalences $C_0(M_n) \sim \mathcal{C}(M_n) \mathbin{\widehat{\otimes}} \mathbb{C}_1$.*

Although $\mathcal{C}(M_n)$ and $\mathcal{A}(M_n)$ carry natural $\mathbb{Z}_2$-gradings, when we consider their C^*-algebra K-theory, we will ignore these gradings. That is, if A is any C^*-algebra—graded or not—then $K_j(A)$ $(j = 0, 1)$ will denote the K-theory group of the underlying C^*-algebra, without the grading. Since C^*-algebra K-theory is Morita invariant, we have the following.

Corollary 5.15. *If M_{2k} is an even-dimensional oriented Riemannian manifold with* spin^c*-structure, there is a canonical K-theory isomorphism*[6]

$$K_j\big(\mathcal{A}(M_{2k})\big) \cong K^{j+1}(M_{2k}).$$

The next result is proved by Trout [45, Theorem 2.14]:

Thom Isomorphism Theorem 5.16. *If $E \to M_n$ is a smooth finite-rank affine Euclidean bundle, then the $*$-homomorphism $\Psi_p : \mathcal{A}(M_n) \to \mathcal{A}(E)$ from Theorem* 2.11 *induces an isomorphism of Abelian groups*:

$$\Psi_* : K_j\big(\mathcal{A}(M_n)\big) \to K_j\big(\mathcal{A}(E)\big), \quad \textit{for } j = 0, 1.$$

In fact, it is the C^*-algebraic formulation of the classical Thom isomorphism $\Phi : K^j(M) \to K^j(E)$ from topological K-theory.

Corollary 5.17. [45, Corollary 2.20] *If E is a finite even-rank oriented Euclidean* spin^c*-bundle* (*with spin connection* ∇) *on an even-dimensional oriented Riemannian* spin^c*-manifold M_n, then $\Psi_p : \mathcal{A}(M_n) \to \mathcal{A}(E)$ induces the topological Thom isomorphism Φ, as depicted in the following commutative diagram*:

$$\begin{tikzcd} K_j(\mathcal{A}(M_n)) \arrow[r, "\Psi_*"] \arrow[d, "\cong"'] & K_j(\mathcal{A}(E)) \arrow[d, "\cong"] \\ K^{j+1}(M_n) \arrow[r, "\Phi"] & K^{j+1}(E) \end{tikzcd}$$

Although the connecting maps $\alpha_n : \mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})$ are not functorial at the C^*-algebra level (as in diagram (1)), they are at the level of K-theory.

Lemma 5.18. *The following diagram of Abelian groups*

$$\begin{tikzcd} & K_j(\mathcal{A}(M_{n+1})) \arrow[dr, "(\alpha_{n+1})_*"] & \\ K_j(\mathcal{A}(M_n)) \arrow[ur, "(\alpha_n)_*"] \arrow[rr, "(\alpha_n^{n+2})_*"'] & & K_j(\mathcal{A}(M_{n+2})) \end{tikzcd} \tag{12}$$

[6] It is also true that $K_j(\mathcal{C}(M_{2k})) \cong K^j(M_{2k})$, but we shall not use this here.

commutes for all $n \geqslant k$ and $j = 0, 1$, where $\alpha_n^{n+2} : \mathcal{A}(M_n) \to \mathcal{A}(M_{n+2})$ is any Gysin map induced by the inclusion $M_n \hookrightarrow M_{n+2}$ (as in Diagram (6)).

Proof. The functor $M_n \mapsto K_j(\mathcal{A}(M_n))$ from the category of finite-dimensional smooth (Riemannian) manifolds is homotopy-invariant, has Gysin maps (independent of the choice of tubular neighborhood) and, most importantly, a *transitive* Thom homomorphism [45, Lemma 3.10]. The result now follows from the corresponding proof in Karoubi [27, Propositions 5.22 and 5.24] for topological K-theory. □

We now come to the main result of our paper.

Theorem 5.19. *Let M be a smooth Fredholm* Spin_q*-manifold with Riemannian metric g and augmented Fredholm filtration $\mathcal{F} = (M_n, U_n)_{n=k}^{\infty}$. With a dimension shift, the K-theory of $\mathcal{A}(M, g, \mathcal{F})$ coincides with the topological K-theory of $(M, \mathcal{F})$ and the (compactly supported) K-homology of M:*

$$K_{j+1}\big(\mathcal{A}(M, g, \mathcal{F})\big) \cong K^{\infty-j}(M, \mathcal{F}) \cong K_j^c(M).$$

Proof. Indeed, using the fact that $2\mathbb{Z}$ is cofinal in $\mathbb{Z}$, we can restrict to the even-dimensional subsequences in the directed limits under consideration:

$$\begin{aligned}
K^{\infty-j}(M, \mathcal{F}) &= \varinjlim K^{n-j}(M_n) && \text{(Definition 5.1)}\\
&\cong \varinjlim K^{2n-j}(M_{2n}) && \text{(cofinal property of direct limits)}\\
&\cong \varinjlim K^{j+2}(M_{2n}) && \text{(Bott periodicity)}\\
&\cong \varinjlim K_{j+1}\big(\mathcal{A}(M_{2n})\big) && \text{(Corollary 5.15)}\\
&\cong \varinjlim K_{j+1}\big(\mathcal{A}(M_n)\big) && \text{(cofinal property of direct limits)}\\
&\cong K_{j+1}\big(\varinjlim \mathcal{A}(M_n)\big) && \text{(continuity of } K\text{-theory)}\\
&= K_{j+1}\big(\mathcal{A}(M, g, \mathcal{F})\big) && \text{(Definition 4.1).} \qquad \square
\end{aligned}$$

As the compactly supported K-homology of M does not depend on the metric and on the choice of augmented filtration, we get in particular the following independence on the metric and the filtration (compare again with [35, Theorem 2.1]):

Corollary 5.20. *If M is a smooth Fredholm* Spin_q*-manifold, as above, then its topological K-theory $K^{\infty-j}(M, \mathcal{F})$ and the K-theory of $\mathcal{A}(M, g, \mathcal{F})$ do not depend on the choices of the metric g and augmented Fredholm filtration $\mathcal{F}$.*

Another easy consequence is:

Corollary 5.21. *If $\mathcal{E}$ is a separable infinite-dimensional Euclidean space, then*

$$K_j\big(\mathcal{A}(\mathcal{E})\big) \cong K_j\big(\mathcal{A}(S_{\mathcal{E}})\big) \cong \begin{cases} 0 & \text{if } j = 0,\\ \mathbb{Z} & \text{if } j = 1 \end{cases}$$

where $S_{\mathcal{E}}$ denotes the unit sphere in $\mathcal{E}$.

Acknowledgement

The authors would like to thank John Roe, Carolyn Gordon, David Webb, Dana Williams, Gregory Leibon, and the referee for interesting discussions and helpful suggestions.

References

[1] R. Abraham, J.E. Marsden, T. Ratiu, Manifolds, Tensor Analysis, and Applications, second ed., Appl. Math. Sci., vol. 75, Springer, New York, 1988.

[2] M. Anastasiei, Spin structures on Hilbert manifolds, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) 24 (2) (1978) 367–373.
[3] M.F. Atiyah, K-Theory, W.A. Benjamin, New York, 1967.
[4] M.F. Atiyah, R. Bott, A. Shapiro, Clifford modules, Topology 3 (Suppl. 1) (1964) 3–38.
[5] N. Berline, E. Getzler, M. Vergner, Heat Kernels and Dirac Operators, Grundlehren Math. Wiss., vol. 298, Springer, Berlin, 1986.
[6] B. Blackadar, K-Theory for Operator Algebras, second ed., MSRI Publ. Ser., vol. 5, Springer, New York, 1998.
[7] R. Bott, The stable homotopy of the classical groups, Ann. of Math. 70 (1959) 313–337.
[8] L.G. Brown, R.G. Douglas, P.A. Fillmore, Extensions of C^*-algebras and K-homology, Ann. of Math. (2) 105 (2) (1977) 265–324.
[9] V.M. Buhštaber, A.S. Miščenko, Elements of infinite filtration in K-theory, Dokl. Akad. Nauk SSSR 178 (1968) 1234–1237.
[10] V.M. Buhštaber, A.S. Miščenko, A K-theory on the category of infinite cell complexes, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968) 560–604.
[11] A. Connes, Noncommutative Geometry, Academic Press, New York, 1994.
[12] A. Connes, G. Skandalis, The longitudinal index theorem for foliations, Publ. Res. Inst. Math. Sci. 20 (6) (1984) 1139–1183.
[13] P. de la Harpe, Classical Banach–Lie algebras and Banach–Lie Groups of Operators in Hilbert Space, Lecture Notes in Math., vol. 285, Springer, Berlin, 1972.
[14] P. de la Harpe, The Clifford algebra and the spinor group of a Hilbert space, Compositio Math. 25 (1972) 245–261.
[15] P. de la Harpe, Some properties of infinite-dimensional orthogonal groups, in: Global Analysis and Its Applications, Internat. Atomic Energy Agency, Vienna, 1974, pp. 71–77.
[16] J. Dixmier, Champs continus d'espaces hilbertiens et de C^*-algebras, J. Math. Pure Appl. 42 (1993) 1–20.
[17] E. Dyer, Cohomology Theories, Mathematics Lecture Note Series, W.A. Benjamin, New York, 1969.
[18] J. Eells, Fredholm structures, in: Nonlinear Functional Analysis, Part 1, Chicago, IL, 1968, in: Proc. Sympos. Pure Math., vol. XVIII, American Mathematical Society, Providence, RI, 1970, pp. 62–85.
[19] J. Eells, K.D. Elworthy, Open embeddings of certain Banach manifolds, Ann. of Math. (2) 91 (1970) 465–485.
[20] J. Eells, K.D. Elworthy, On Fredholm manifolds, in: Actes du Congrès International des Mathématiciens, Nice, 1970, Tome 2, Gauthier-Villars, Paris, 1971, pp. 215–219.
[21] K.D. Elworthy, A.J. Tromba, Differential structures and Fredholm maps on Banach manifolds, in: Global Analysis, Berkeley, CA, 1968, in: Proc. Sympos. Pure Math., vol. XV, American Mathematical Society, Providence, RI, 1970, pp. 45–94.
[22] D.S. Freed, The geometry of loop groups, J. Differential Geom. 28 (2) (1988) 223–276.
[23] J.M. Gracia-Bondía, J.C. Varilly, H. Figueroa, Elements of Noncommutative Geometry, Birkhäuser Advanced Texts, Birkhäuser, Boston, 2001.
[24] N. Higson, G. Kasparov, E-theory and K-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. 144 (2001) 23–74.
[25] N. Higson, G. Kasparov, J. Trout, A bott periodicity theorem for infinite-dimensional euclidean space, Adv. Math. 135 (1) (1998) 1–40.
[26] N. Higson, J. Roe, Analytic K-Homology, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.
[27] M. Karoubi, K-Theory, An Introduction, Grundlehren Math. Wiss., vol. 226, Springer, New York, 1978.
[28] G.G. Kasparov, The operator K-functor and extensions of C^*-algebras, Math. USSR Izv. 16 (1981) 513–572.
[29] G.G. Kasparov, Equivariant KK-theory and the Novikov Conjecture, Invent. Math. 91 (1988) 147–201.
[30] U. Koschorke, Infinite-dimensional K-theory and characteristic classes of Fredholm bundle maps, in: Global Analysis, Berkeley, CA, 1968, in: Proc. Sympos. Pure Math., vol. XV, American Mathematical Society, Providence, RI, 1970, pp. 95–133.
[31] N.H. Kuiper, The homotopy type of the unitary group of Hilbert space, Topology 3 (1965) 19–30.
[32] S. Lang, Differential and Riemannian Manifolds, third ed., Graduate Texts in Math., vol. 160, Springer, New York, 1995.
[33] H.B. Lawson, M.-L. Michelsohn, Spin Geometry, Princeton University Press, Princeton, NJ, 1989.
[34] K.K. Mukherjea, The homotopy type of Fredholm manifolds, Trans. Amer. Math. Soc. 149 (1970) 653–663.
[35] K.K. Mukherjea, The algebraic topology of Fredholm manifolds, in: Analyse Globale, Sém. Math. Supérieures, No. 42, Univ. Montréal, Montreal, PQ, 1969, Presses University of Montréal, Montreal, PQ, 1971, pp. 163–177.
[36] R.S. Palais, On the homotopy type of certain groups of operators, Topology 3 (1965) 271–279.
[37] N.C. Phillips, Inverse limits of C^*-algebras, J. Operator Theory 19 (1) (1988) 159–195.
[38] N.C. Phillips, Representable K-theory for σ-C^*-algebras, K-Theory 3 (5) (1989) 441–478.
[39] R. Plymen, Strong Morita equivalence, spinors and symplectic spinors, J. Operator Theory 16 (1986) 305–324.
[40] R.J. Plymen, R.F. Streater, A model of the universal covering group of $\mathrm{SO}(H)_2$, Bull. London Math. Soc. 7 (3) (1975) 283–288.
[41] I. Raeburn, D.P. Williams, Pull-backs of C^*-algebras and crossed products by certain diagonal actions, Trans. Amer. Math. Soc. 287 (1985) 755–777.
[42] I. Raeburn, D.P. Williams, Morita Equivalence and Continuous-Trace C^*-Algebras, Math. Surveys Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998.
[43] A. Rennie, Smoothness and locality for nonunital spectral triples, K-Theory 28 (2) (2003) 127–165.
[44] M.A. Rieffel, Morita equivalence for operator algebras, Proc. Sympos. Pure Math. 38 (1982) 285–298.
[45] J. Trout, A Thom isomorphism for infinite rank Euclidean bundles, Homology Homotopy Appl. 5 (2003) 121–159.
[46] N.E. Wegge-Olsen, K-Theory and C^*-Algebras, Oxford University Press, New York, 1993.
[47] J. Weidner, KK-groups for generalized operator algebras. I, K-Theory 3 (1) (1989) 57–77;
J. Weidner, KK-groups for generalized operator algebras. II, K-Theory 3 (1) (1989) 79–98.
[48] G.W. Whitehead, Elements of Homotopy Theory, Graduate Texts in Math., vol. 61, Springer, New York, 1978.
[49] G. Yu, The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Invent. Math. 139 (1) (2000) 201–240.